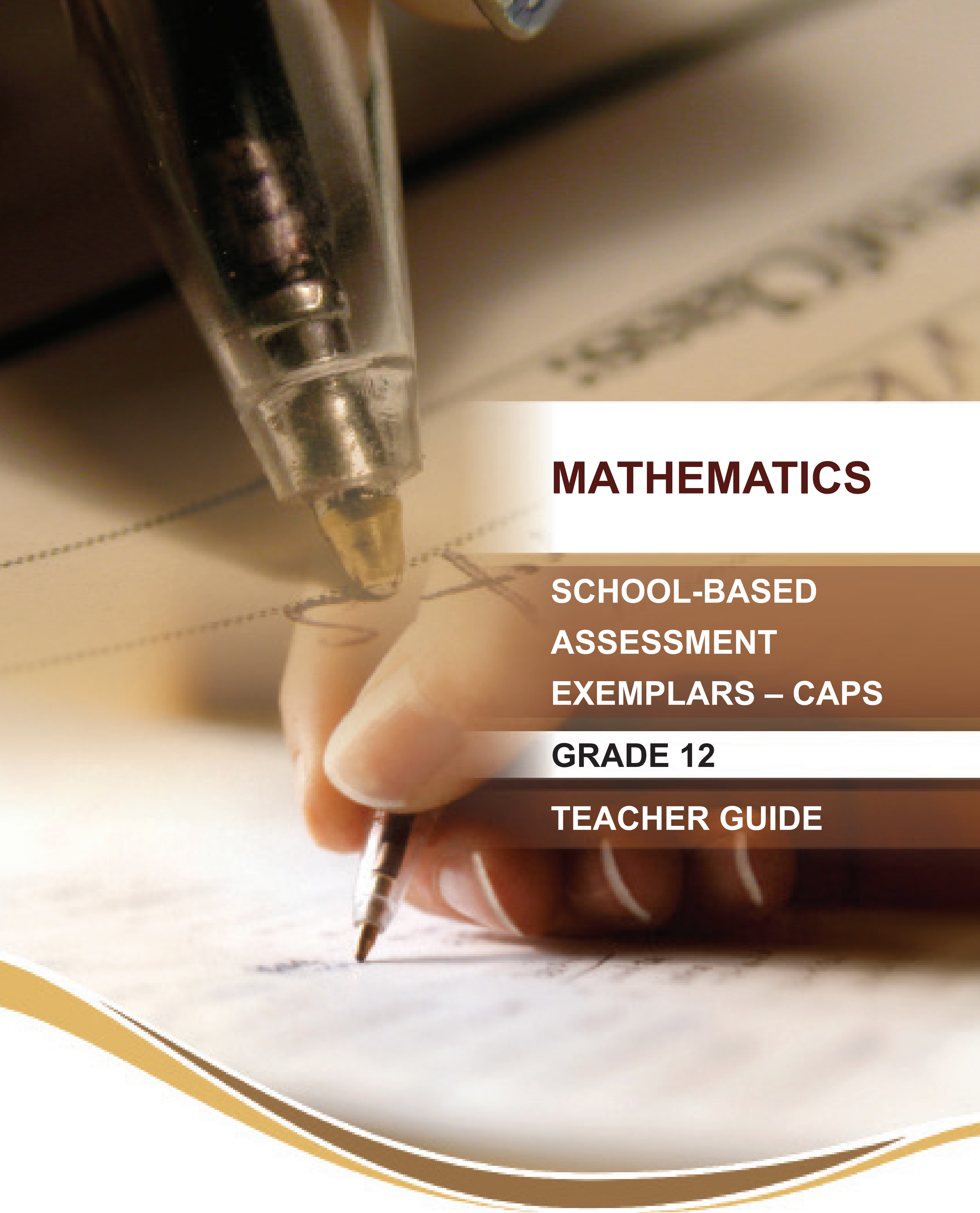

MATHEMATICS

SCHOOL-BASED ASSESSMENT EXEMPLARS – CAPS

GRADE 12

TEACHER GUIDE

basic education

Department:
Basic Education
REPUBLIC OF SOUTH AFRICA

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CONTENTS

1. INTRODUCTION

Assessment is a continuous, planned process using various forms of tasks in order to identify, gather and interpret information about the performance of learners. It involves four steps: generating and collecting evidence of achievement, evaluating this evidence, recording the findings and using this information to understand and assist in the learner's development in order to improve the process of learning and teaching. Assessment should be both informal (assessment for learning) and formal (assessment of learning). In both cases regular feedback should be provided to learners to enhance the learning experience.

2. AIMS AND OBJECTIVES

The purpose of this document is to provide both educators and learners with a set of benchmarked school-based assessment (SBA) tasks. It contains useful information and guidelines in the form of exemplars.

The aim of assessment for teaching and learning is to collect information on a learner's achievement which can be used to improve individual learning. The DBE embarked on a nationwide moderation process of SBA tasks, and it was discovered during this process that many schools across the country do not follow the requirements and guidelines when setting tasks, particularly the investigation and assignment; hence these exemplars were developed to be used by educators as a guide when developing their own tasks.

3. ASSESSMENT TASKS

Although assessment guidelines are included in the annual teaching plan at the end of each term, the following general principles apply:

Tests and examinations are usually time-limited and assessed using a marking memorandum.

Assignments are generally extended pieces of work in which time constraints have been relaxed and which may be completed at home. Assignments may be used to consolidate or deepen understanding of work done earlier. They may thus consist of a collection of past examination questions or innovative activities using any resource material. It is, however, advised that assignments be focused.

Projects are more extended tasks that may serve to deepen understanding of curricular mathematics topics. They may also involve extracurricular mathematical topics where the learner is expected to select appropriate mathematical content to solve context-based or real-life problems. The focus should be on the mathematical concepts and not on duplicated pictures and regurgitation of facts from reference material.

Investigations are set to develop the mathematical concepts or skills of systematic investigation into special cases with a view to observing general trends, making conjectures and proving them. It is recommended that while the initial investigation can be done at home, the final write-up be done in the classroom, under supervision and without access to any notes. Investigations are marked using a rubric which can be specific to the task, or generic, listing the number of marks awarded to each skill as outlined below:

- 40% for communicating individual ideas and discoveries, assuming the reader has not come across the task before. The appropriate use of diagrams and tables will enhance the assignment, investigation or project.
- 35% for generalising, making conjectures and proving or disproving these conjectures;
- 20% for the effective consideration of special cases; and
- 5% for presentation, that is, neatness and visual impact.

4. PROGRAMME OF ASSESSMENT

All assessment tasks that make up a formal programme of assessment for the year are regarded as formal assessment. Formal assessment tasks are marked and formally recorded by the teacher for progress and certification purposes. All formal assessment tasks are subject to moderation for purposes of quality assurance. Generally, formal assessment tasks provide teachers with a systematic way of evaluating how well learners are progressing in a grade and/or a particular subject. Examples of formal assessment tasks include tests, examinations, practical tasks, projects, oral presentations, demonstrations, performances, etc. Formal assessment tasks form part of a year-long formal programme of assessment in each grade and subject.

Formal assessment tasks in mathematics include tests, mid-year examination, preparatory examination (Grade 12), an assignment, a project or an investigation. The forms of assessment used should be appropriate to the age and developmental level of learners. The design of these tasks should cover the content of the subject and include a variety of activities designed to achieve the objectives of the subject. Formal assessment tasks need to accommodate a range of cognitive levels and abilities of learners as indicated in the FET curriculum and assessment policy statement (CAPS).

Informal assessment involves daily monitoring of a learner's progress. This can be done through observations, discussions, practical demonstrations, learner-teacher conferences, informal classroom interactions, etc. Informal assessment may be as simple as stopping during the lesson to observe learners or to discuss with learners how learning is progressing. Informal assessment should be used to provide feedback to learners and to inform planning for teaching and learning, and it need not be recorded. This should not be seen as separate from learning activities taking place in the classroom. Learners or teachers can evaluate these tasks. Self-assessment and peer assessment actively involve learners in assessment. Both are important as they allow learners to learn from and reflect on their own performance. The results of the informal daily assessment activities are not formally recorded, unless the teacher wishes to do so. The results of informal daily assessment tasks are not taken into account for promotion and/or certification purposes.

5. QUALITY ASSURANCE PROCESS

A team of experts comprising teachers and subject advisors from different provinces was appointed by the DBE to develop and compile the assessment tasks in this document. The team was required to extract excellent pieces of learner tasks from their respective schools and districts. This panel of experts spent a period of four days at the DBE developing tasks based on guidelines and policies. Moderation and quality assurance of the tasks were undertaken by national and provincial examiners and moderators. The assessment tasks were further refined by the national internal moderators to ensure that they are in line with CAPS requirements.

1. ASSIGNMENT: SEQUENCES AND SERIES **TOTAL: 60**

INSTRUCTIONS

1. Answer all the questions.
2. Clearly show all calculations you have used in determining your answers.
3. Round answers off to TWO decimal places, unless stated otherwise.
4. Number your answers correctly according to the numbering system used in this question paper.
5. Write neatly and legibly.

QUESTION 1

Lucy is arranging 1-cent and 5-cent coins in rows. The pattern of the coins in each row is shown below.

1.1	Calculate the total number of coins in the 40^{th} row.	(3)
1.2	Calculate the total value of the coins in the 40^{th} row.	(4)
1.3	Which row has coins with a total value of 337 cents?	(6)
1.4	Show that the total value of the coins in the first 40 rows is 4 800 cents.	(6)

[19]

QUESTION 2

The sum of the first n terms of a sequence is given by: $S_n = n(23 - 3n)$

2.1 Write down the first THREE terms of the sequence. (5)
2.2 Calculate the 15^{th} term of the sequence. (3)
[8]

QUESTION 3

The sum of the second and third terms of a geometric sequence is 280, and the sum of the fifth and the sixth terms is 4 375. Determine:

3.1 The common ratio AND the first term. (6)
3.2 The sum of the first 10 terms. (2)
[8]

QUESTION 4

Determine the value of k if:

$$\sum_{t=1}^{\infty} 4.k^{t-1} = 5$$ **[6]**

QUESTION 5

Given the series: $2(5)^5 + 2(5)^4 + 2(5)^3 + ...$

Show that this series converges. [2]

QUESTION 6

If 2; x; 18; ... are the first three terms of a geometric sequence, determine the value(s) of x. **[4]**

QUESTION 7

Given: $T_n = 3^{n+1}$. Which term is the first to exceed 20 000? **[4]**

QUESTION 8

The sequence 3; 9; 17; 27; ... is a quadratic sequence.

8.1 Write down the next term of the sequence. (1)
8.2 Determine an expression for the n^{th} term of the sequence. (4)
8.3 What is the value of the first term of the sequence that is greater than 269? (4)
[9]

TOTAL:60

2. INVESTIGATION 1: FUNCTIONS AND INVERSES

TOTAL: 50

INSTRUCTIONS

1. Answer all the questions.
2. Clearly show all calculations you have used in determining your answers.
3. Round answers off to TWO decimal places, unless stated otherwise.
4. Number your answers correctly according to the numbering system used in this question paper.
5. Write neatly and legibly.

PART 1: WHICH RELATIONS CONSTITUTE FUNCTIONS?

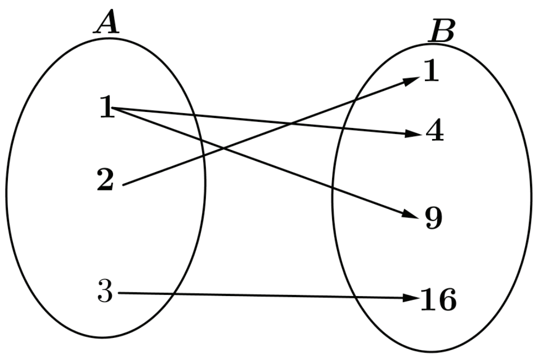
One-to-one relation: A relation is one-to-one if for every input value there is only one output value.

Many-to-one relation: A relation is many-to-one if for more than one input value there is one output value.

One-to-many relation: A relation is one-to-many if for one input value there is more than one output value.

1.1 Determine the type of relation in each case and give a reason.

1.1.1

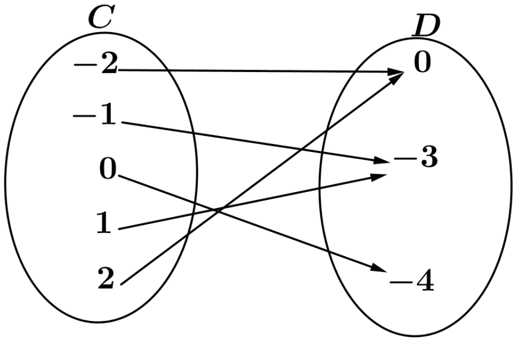


(1)

1.1.2 $\{(1 ; 3), (2 ; 5), (6 ; 13), (7 ; 15)\}$

(1)

1.1.3



____ (1)

____

A function is a set of ordered number pairs where no two ordered pairs have the same x-coordinate, or put differently: a function is a set of ordered pairs where, for every value of x there is one and only one value for y. However, for the same value of y there may be different values for x.

1.2 Which of the relations (in QUESTIONS 1.1.1 to 1.1.3) are functions? Why?

(a)____ (2)

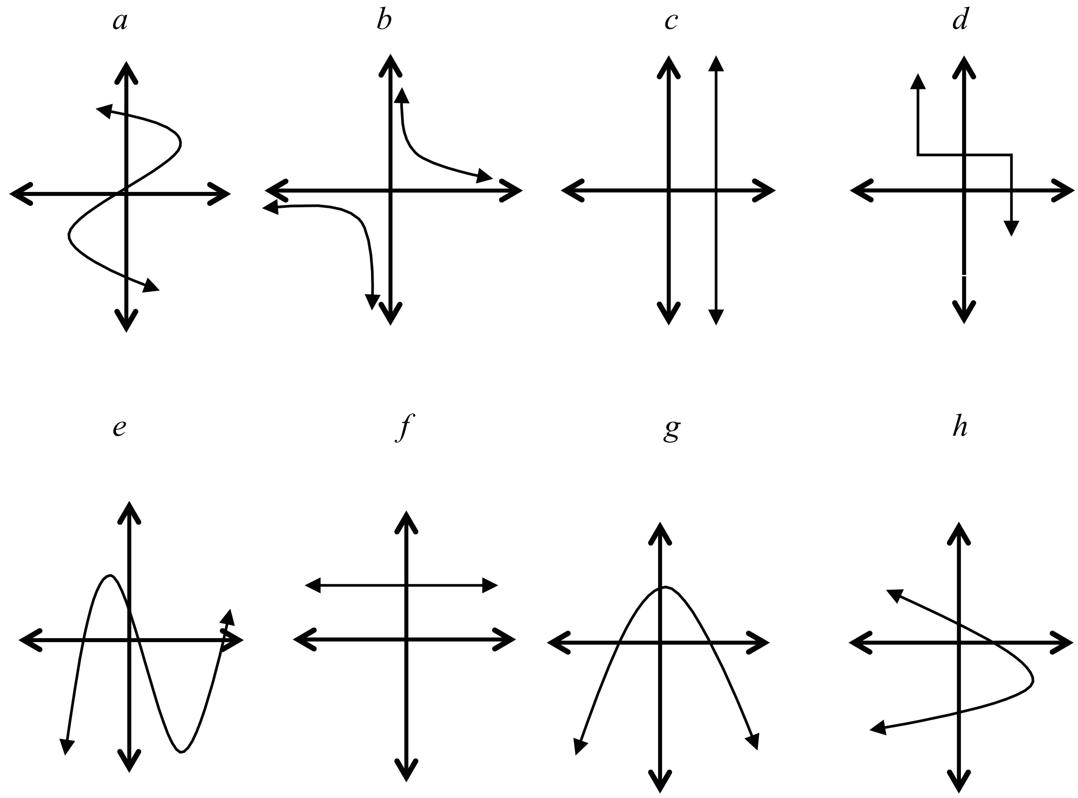
(b)____ (1)

(c)____ (1)

The vertical-line test is used to determine whether or not a given graph is a function.

To determine whether a graph is a function, do the vertical-line test. If any vertical line intersects the graph of f only once, then f is a function; and if any vertical line intersects the graph of f more than once, then f is not a function.

1.3 Determine whether or not the following graphs are functions. Give a reason for your answer.



(a)____ (1)

(b)____ (1)

(c)____ (1)

(d)____ (1)

(e)____ (1)

(f)____ (1)

(g)____ (1)

(h)____ (1)

PART 2: THE INVERSE OF AN EXPONENTIAL FUNCTION

2.1 Consider the equation $g(x) = 2^x$. Now complete the following table:

x	-3	-2	-1	0	1	2
y						

(1)

2.2 Sketch the graph of g.

2.3 Sketch the graph of $f(x) = x$ as a dotted line on the same set of axes as g. (1)

2.4 Complete the table below for h, if h is g when the x and y values are interchanged.

x						
y						

Draw h on the same set of axes as g. (4)

2.5 Hence, write down the x-intercept of each of the following graphs below.

$y = 2^x$ $x = 2^y$

2.5.1____

2.5.2____ (2)

2.6 Write down the domain and range of:

2.6.1 $y = 2^x$

Domain: ___________________

Range: ____________________ (2)

2.6.2 $x = 2^y$

Domain:____________________

Range :_____________________ (2)

2.6.3 What is the relationship between the domain and the range of the two graphs in 2.6.1 and 2.6.2

____

____ (1)

2.6.4 Are both graphs functions? Give a reason for your answer .

____ (2)

2.6.5 Write the equation of $x = 2^y$ in the form $y =$

____ (1)

2.6.6 Do you notice any line of symmetry in your sketch? What is the equation of this line?

____ (1)

2.6.7 In mathematics we call h the inverse of g. Make a conjecture about the graph and its inverse.

____ (3)

PART 3: WHEN IS THE INVERSE OF A QUADRATIC FUNCTION ALSO A FUNCTION?

3.1 Given: $f(x) = 2x^2$, for $x \in \mathbb{R}$

(1)

3.1.1 Write down the equation of the inverse of f.

____

3.1.2 Write down the turning points of both f and its inverse.

____ (2)

3.1.3 Sketch the graphs of f and its inverse on the same set of axes.

(2)

3.1.4 Decide whether or not the inverse of f is a function, and give a reason for your answer.

(2)

____

3.1.5 Explain how you would restrict the domain of f such that its inverse is a function.

____ (2)

3.1.6 Hence, write down the corresponding range of the inverse of f if:

(a) $x \leq 0$ ____ (1)

(b) $x \geq 0$ ____ (1)

3.1.7 On separate sets of axes, sketch the graphs of the inverse of f with restricted domains as in QUESTION 3.1.6. Indicate the domain and range of each.

(2)

3.1.8 Are the two graphs in QUESTION 3.1.7 functions? Give a reason or reasons for your answer.

____ (2)

TOTAL: 50

3. INVESTIGATION 2: APPLICATION OF DIFFERENTIAL CALCULUS

TOTAL: 50

INSTRUCTIONS

1. Answer all the questions.
2. Clearly show all calculations you have used in determining your answers.
3. Round answers off to TWO decimal places, unless stated otherwise.
4. Number your answers correctly according to the numbering system used in this question paper.
5. Write neatly and legibly.

OBJECTIVE: Investigating the point of inflection of a cubic graph and its relationship with the graphs of the first and the second derivatives.

CASE 1

Given: $f(x) = x^3 - 7x^2 + 36$

1.1	Draw the graph of f neatly on graph paper. Clearly indicate all intercepts and coordinates of turning points.	(8)
1.2	Determine the first derivative of f, and name it g.	(1)
1.3	Draw the graph of g on the same set of axes as f. Clearly show all intercepts and the turning point.	(3)
1.4	Determine the second derivative of f and name it h. Then sketch the graph of h on the same set of axes as f and g. Clearly show all intercepts of the graph with the axes.	(3)
1.5	What do you notice regarding the x-intercepts of the quadratic function and the x-coordinates of the turning points of the cubic function?	(1)
1.6	The point of inflection can be determined by solving $f''(x) = 0$. It can also be determined by calculating the midpoint of the turning points of the cubic graph. Hence, determine the point of inflection of f.	(3)
1.7	What do you notice regarding the axis of symmetry of g, the x-intercept of h and the x-coordinate of the point of inflection of f?	(1)

[20]

CASE 2

Given: $f(x) = -x^3 - 2x^2 + 4x + 8$

2.1	Draw the graph of f neatly on graph paper. Clearly indicate all intercepts and coordinates of turning points.	(7)
2.2	Determine the first derivative of f, and name it g.	(1)
2.3	Draw the graph of g on the same set of axes as f, and clearly show all intercepts and the turning point.	(4)
2.4	Determine the second derivative of f and name it h, then sketch the graph of h on the same set of axes as f and g. Clearly show all intercepts of the graph with the axes.	(3)
2.5	What do you notice regarding the x-intercepts of the quadratic function and the	(1)

x-coordinates of the turning points of the cubic function?

2.6 The point of inflection can be determined by solving $f''(x) = 0$. It can also be determined by calculating the midpoint of the turning points of the cubic graph. Hence, determine the point of inflection of f. (3)

2.7 What do you notice regarding the axis of symmetry of g, the x-intercept of h and the x-coordinate of the point of inflection of f? (1)

[20]

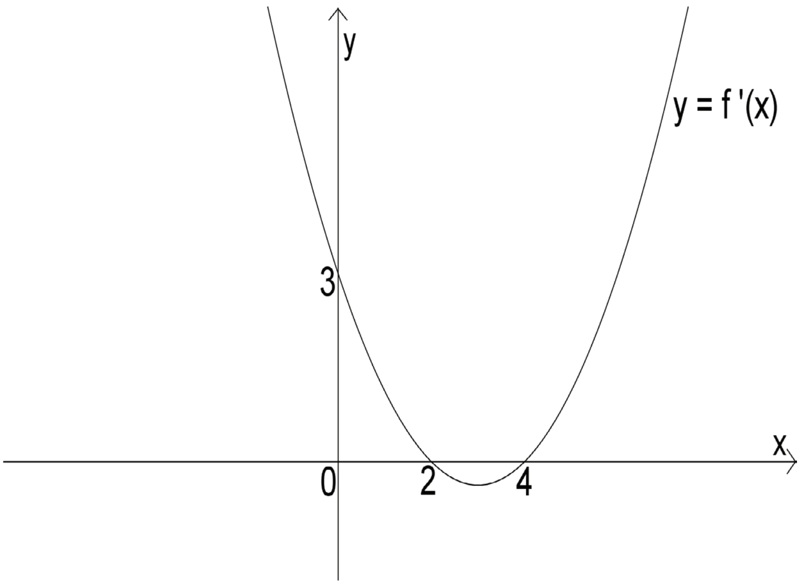
3. CONCLUSION

Based on the two cases, what conclusion can you draw about the point of inflection of a cubic function in relation to the graphs of the first and second derivatives?

[2]

4. APPLICATION

The parabola shown below is the graph of the derivative of a function f.



4.1 For what value(s) of x is f:

4.1.1 Increasing (2)

4.1.2 Decreasing (2)

4.2 Give the abscissae of the turning point(s) of $y = f(x)$. (2)

4.3 Classify the stationary point(s). (2)

[8]

TOTAL: 50

4. PROJECT: A PRACTICAL APPLICATION OF DIFFERENTIAL CALCULUS TOTAL: 50

INSTRUCTIONS

1. Answer all the questions.
2. Clearly show all calculations you have used in determining your answers.
3. Round answers off to TWO decimal places, unless stated otherwise.
4. Number your answers correctly according to the numbering system used in this question paper.
5. Write neatly and legibly,
6. Sketch the containers according to the given specifications.
7. Mathematical methods and formulae need to be used to plan and sketch the containers.
8. All calculations and planning of the side lengths and surface areas must be neatly and clearly presented in writing and sketches.

CONTAINERS

A: A container with a rectangular base

B: A container with a circular base

C: A container with a triangular base

SPECIFICATIONS

- Each container must hold exactly one litre of liquid.
- Each container must have a minimum surface area.
- The surface area of each container must include the lid.
- The length of the rectangular base must be twice the breadth.
- The triangular container must have an equilateral base.

FURTHER COMPARISON

Apart from your conclusion based on the three options, what other shape of soft drink container would you use in the manufacturing of soft drink cans? Give a reason for your answer.

HINT: The shape in question would be the most economical to manufacture but may not be the most practical choice.

RUBRIC

CRITERIA	MAXIMUM MARK	MARKS AWARDED		
		A	B	C
Correct mathematical formulae	3 x 3			
Correct calculations:				
Measurements of bases	4 x 3			
Height of the containers	2 x 3			
Logical reasoning and presentation	3 x 3			
Submitting on time	2			
Conclusion of the least material needed	1 x 3			
Final, further comparison	1 x 3			
Sketches	2 x 3			
TOTAL	**50**			

1. ASSIGNMENT **TOTAL: 60**

MEMORANDUM: SEQUENCES AND SERIES

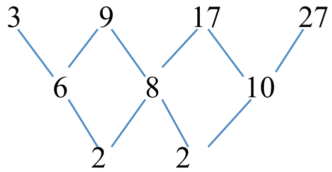
1.1	The sequence below can be used to determine the total number of coins in the 40^{th} row: 1; 3; 5; 7… Arithmetic sequence $a = 1$ and $d = 2$ $T_{40} = ?$ $T_n = a + (n-1)d$ $T_{40} = 1 + (40-1)2$ $= 79$ OR $n = 40$, which is an even number ∴ Number of coins: $T_n = n - 1 + n$ $T_n = 2n - 1$ $T_{40} = 2(40) - 1$ $= 79$	✓ for $d = 2$ ✓substitution in correct formula ✓ answer ✓ $T_n = 2n - 1$ ✓substitution in correct formula ✓ answer (3)
1.2	$n = 40$, which is an even number $\therefore T_n = (n-1)(1) + n(5)$ Total value $= (40-1)(1) + (40)(5)$ $= 39 + 200$ $= 239$ **OR** Even rows form arithmetic sequence: 11; 23; 35; 42… $a = 11; d = 12;\ n = 20$ $T_n = a + (n-1)d$ $T_{20} = 11 + (20-1)(12)$ $= 239$	✓$(n-1)$ ✓$5n$ ✓substitution in correct formula ✓ answer ✓ for sequence ✓ $d = 12$ ✓substitution in correct formula ✓ answer (4)
1.3	If n is odd : $T_n = (n-1)(5) + n(1) = 6n - 5$ $T_n = 337$ $n = ?$ $(n-1)(5) + n(1) = 337$ $5n - 5 + n = 37$ $6n = 342$ $n = 57$ If n is even: $T_n = (n-1)(1) + n(5) = 337$ $6n - 1 = 337$ $6n = 338$ $n = 56.333\ldots$ or $56\frac{1}{3}$ **Not applicable** since $56\frac{1}{3}$ is not a natural number	$(n-1)(5) + n(1) = 337$ ✓ equation ✓simplifying ✓ answer $(n-1)(1) + n(5) = 337$ ✓ equation ✓ answer ✓ not applicable (6)

1.4	$S_{40} = 1 + 11 + 13 + 23 + 25 + 35 + \cdots + 239$ $= (1 + 13 + 25 + \cdots) + (11 + 23 + 35 + \cdots)$ $= \frac{20}{2}[2 + (20-1)12] + \frac{20}{2}[22 + (20-1)2]$ $= 10(230) + 10(250)$ $= 4\,800 \text{ cents}$ OR Series for 1-cent coins: $S_{40} = 1 + 1 + 3 + 3 + 5 + 5 + \cdots + 39 + 39$ $= 2(1 + 3 + 5 + \cdots)$ $= 2S_{20}$ $a = 1, d = 2, n = 20$ $S_n = \frac{n}{2}[2a + (n-1)d]$ $S_{40} = 2\left[\frac{20}{2}(2 + (20-1)2)\right]$ $= 800 \text{ coins}$ Series for 5-cent coins : $S_{40} = 0 + 2 + 2 + 4 + 4 + 6 + 6 + \cdots + 38 + 40$ $= (0 + 2 + 4 + 6 + \cdots) + (2 + 4 + 6 + \cdots)$ Note: This is a combination of two arithmetic series: $0 + 2 + 4 + 6 + \cdots$; $a = 0$ and $d = 2$ $n = 20$ $2 + 4 + 6 + 8 + \cdots$; $a = 2$ and $d = 2$ $n = 2$ $\therefore S_{40} = \frac{20}{2}[0 + (20-1)2] + \frac{20}{2}[4 + (20-1)2]$ $= 380 + 420$ $= 800 \text{ coins}$ Total value is : $800(1) + 800(5) = 4\,800 \text{ cents}$	✓ for generating sequence ✓$(1 + 13 + 25 + \cdots)$ ✓ $11 + 23 + 35 + \cdots$ ✓$\frac{20}{2}[2 + (20-1)12]$ ✓$\frac{20}{2}[22 + (20-1)2]$ ✓ answer ✓ $2(1 + 3 + 5 + \cdots)$ ✓ 800 coins ✓ for generating sequence ✓ for splitting sequence ✓ 800 ✓4 800 (6)

2.		
2.1	$S_n = n(23 - 3n)$ $S_1 = 1[23 - 3(1)]$ $S_1 = 20$ $T_1 = 20$ $S_2 = 2[23 - 3(2)]$ $S_2 = 34$ $T_2 = S_2 - S_1$ $= 34 - 20$ $T_2 = 14$ $S_3 = 3[23 - 3(3)]$ $S_3 = 42$ $T_3 = 42 - 34$ $= 8$	✓substitution in formula $S_1 = 1[23 - 3(1)]$ ✓value $T_1 = 20$ ✓value $S_2 = 34$ ✓value $T_2 = 14$ ✓value $T_3 = 8$ (5)
2.2	$20; 14; 8; \ldots$ $a = 20; d = -6$ The sequence is arithmetic. $T_n = a + (n-1)d$ $T_{15} = 20 + (14)(-6)$ $T_{15} = -64$	✓value of *d* ✓substitution in the correct formula ✓answer (3)
3.		
3.1	$ar^2 + ar = 280$……………(1) $ar^4 + ar^5 = 4375$…………(2) $\frac{(1)}{(2)}$: $\frac{ar^4+ar^5}{ar+ar^2} = \frac{4375}{280}$ $\frac{(1)}{(2)}$: $\frac{ar^4(1+r)}{ar(1+r)} = \frac{4375}{280}$ $r^3 = \frac{125}{8}$ $r = \frac{5}{2}$	✓$ar^2 + ar = 280$ ✓ $ar^4 + ar^5 = 4375$ ✓ $\frac{ar^4 + ar^5}{ar + ar^2} = \frac{4375}{280}$ ✓ for common factor ✓$r^3 = \frac{125}{8}$ ✓ answer (6)

3.2	$ar^2 + ar = 280$ $a\left[\left(\frac{5}{2}\right)^2 + \frac{5}{2}\right] = 280$ $a\left(\frac{25}{4} + \frac{5}{2}\right) = 280$ $a\left(\frac{35}{4}\right) = 280$ $a = 32$	✓$a\left[\left(\frac{5}{2}\right)^2 + \frac{5}{2}\right] = 280$ Substitution of the value of r in equation (1) or (2) ✓ answer (2)
3.3	$S_n = \frac{a(r^n - 1)}{r - 1}$ $a = 32, \quad r = \frac{5}{2}$ or $2,5$ $S_{10} = \frac{32\left[\left(\frac{5}{2}\right)^{10} - 1\right]}{2,5 - 1}$ $= 203429,19$	✓ substitution in the correct formula ✓ answer (2)
4.	$\sum_{r=1}^{\infty} 4.k^{r-1} = 5$ $4+4k + 4k^2 + 4k^3 \ldots = 5$ $S_\infty = \frac{a}{1 - r}$ $a = 4, \quad r = k, \quad S_\infty = 5$ $5 = \frac{4}{1 - k}$ $5 - 5k = 4$ $5k = 1$ $k = \frac{1}{5}$	✓for the series ✓ equating the series to 5 ✓$r = k$ ✓$5 = \frac{4}{1 - k}$ substitution in the formula ✓simplifying ✓ answer (6)

5.	$2(5)^5+2(5)^4+2(5)^3+\cdots$ $r=\frac{T_2}{T_1}$ $r=\frac{2(5)^4}{2(5)^5}$ $r=\frac{1}{5}$ For convergence $-1<r<1$ Since : $-1<\frac{1}{5}<1$ This implies that the series converges.	$r=\frac{1}{5}$ ✓ $-1<\frac{1}{5}<1$ (2)
6.	$2;x;18$ $\frac{x}{2}=\frac{18}{x}$ $x^2=36$ $x^2-36=0$ $(x-6)(x+6)=0$ $x=6$ or $x=-6$ **OR** $\frac{x}{2}=\frac{18}{x}$ $x^2=36$ $x=\pm\sqrt{36}$ $x=\pm6$	✓ $\frac{x}{2}=\frac{18}{x}$ ✓ $x^2=36$ ✓ for both factors ✓ for both values of x (4) ✓ $\frac{x}{2}=\frac{18}{x}$ ✓ $x^2=36$ ✓ $x=\pm\sqrt{36}$ $x=\pm6$ ✓ 1 mark for both values of x

7.	$3^{n+1} > 20\ 000$ $n+1 > log_3 20\ 000$ $n+1 > \frac{\log 20\ 000}{\log 3}$ $n+1 > 9{,}0145\ ....$ $n > 8{,}0145\ ...$ $n = 9$	✓$3^{n+1} > 20\ 000$ For the inequality ✓log form ✓$n+1 > 9{,}0145$ Value of log $n > 8{,}0145$✓ Simplifying ✓answer
8.		
8.1	39	✓answer (1)
8.2	 $2a = 2$ $a = 1$ $c = 3 - 4 = -1$ $T_n = n^2 + bn - 1$ $3 = (1)^2 + b(1) - 1$ $b = 3$ $T_n = n^2 + 3n - 1$ **OR** $T_n = an^2 + bn + c$ $2a = 2$ $a = 1$ $3a + b = 6$ $3(1) + b = 6$ $b = 3$ $a + b + c = 3$ $1 + 3 + c = 3$ $c = -1$ $T_n = n^2 + 3n - 1$	✓$a = 1$ ✓$c = -1$ ✓formula ✓ $b = 3$ ✓formula ✓$a = 1$ ✓ $b = 3$ ✓$c = -1$
8.3	$n^2 + 3n - 1 > 269$ $n^2 + 3n - 270 > 0$ $(n+18)(n-15) > 0$ The first value of n is 16 The term is $16^2 + 3(16) - 1 = 303$	✓$n^2 + 3n - 1 > 269$ ✓factors ✓ $n = 16$ ✓ answer

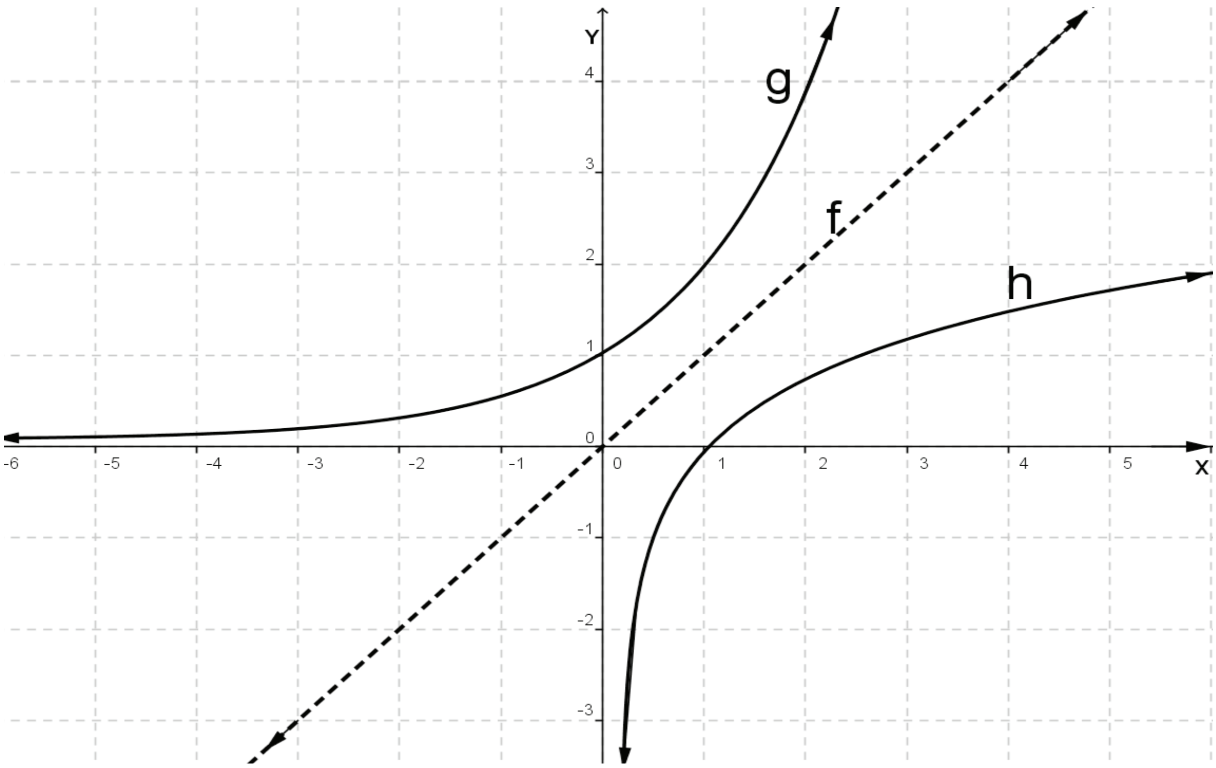
2. INVESTIGATION 1 TOTAL: 50

MEMORANDUM: FUNCTIONS AND INVERSES

PART 1		
1.1		
1.1.1	One-to-many relation	✓answer (1)
1.1.2	One-to-one relation	✓answer (1)
1.1.3	Many-to-one relation	✓answer (1)
1.2		
a)	Not a function, for one input-value there are more than one output values.	✓answer ✓reason (2)
b)	Function, for one input value there is only one output-value.	✓answer and reason (1)
c)	Function, for more than one input value there is one output-value.	✓answer and reason (1)
1.3	a: Not a function b: Function c: Not a function d: Not a function e: Function f: Function g: Function h: Not a function	✓one mark for each answer (8)
PART 2		
2.1	(table below)	✓ one mark for all y-values (1)

x	-3	-2	-1	0	1	2
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4

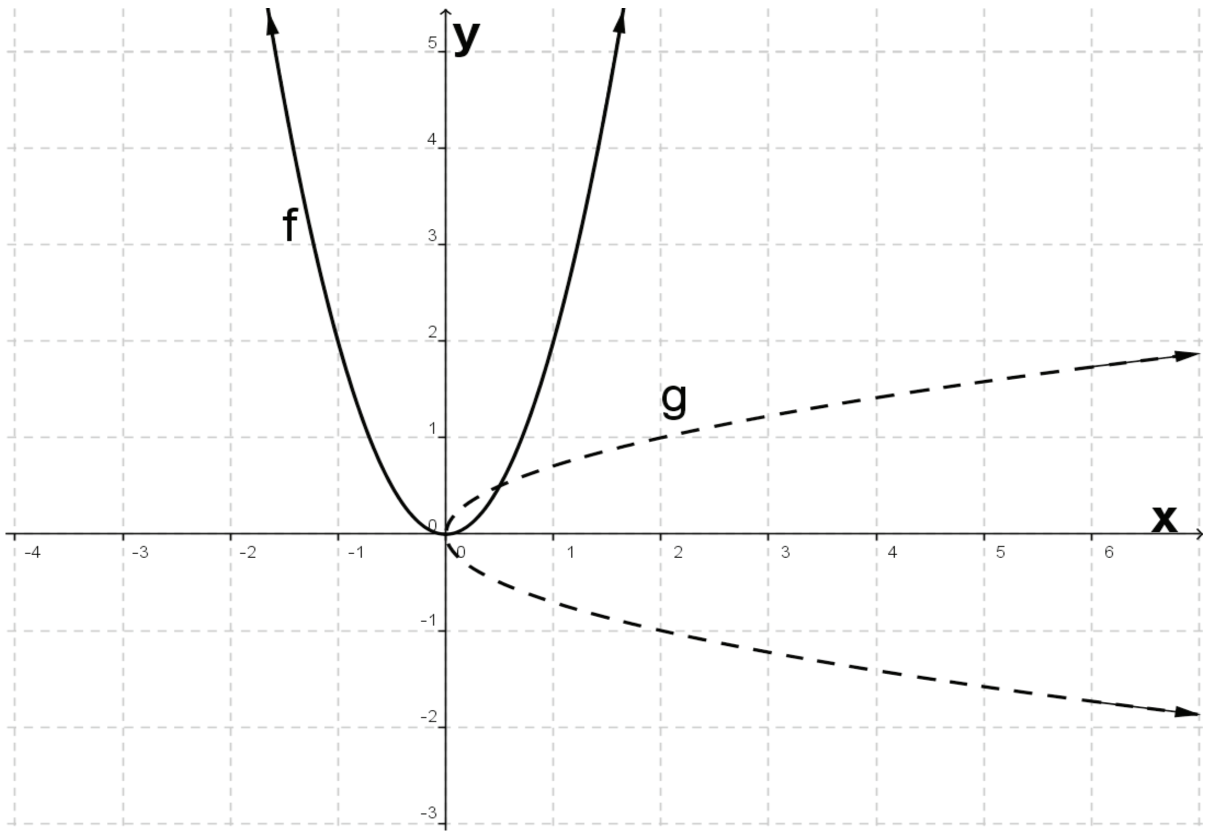
2.2		*g*: ✓ *y*-intercept ✓ shape ✓ asymptote
2.3		*f*: ✓ for both *x*- and *y*-intercepts *h*: ✓ for *x*-intercept ✓ for asymptote



(6)

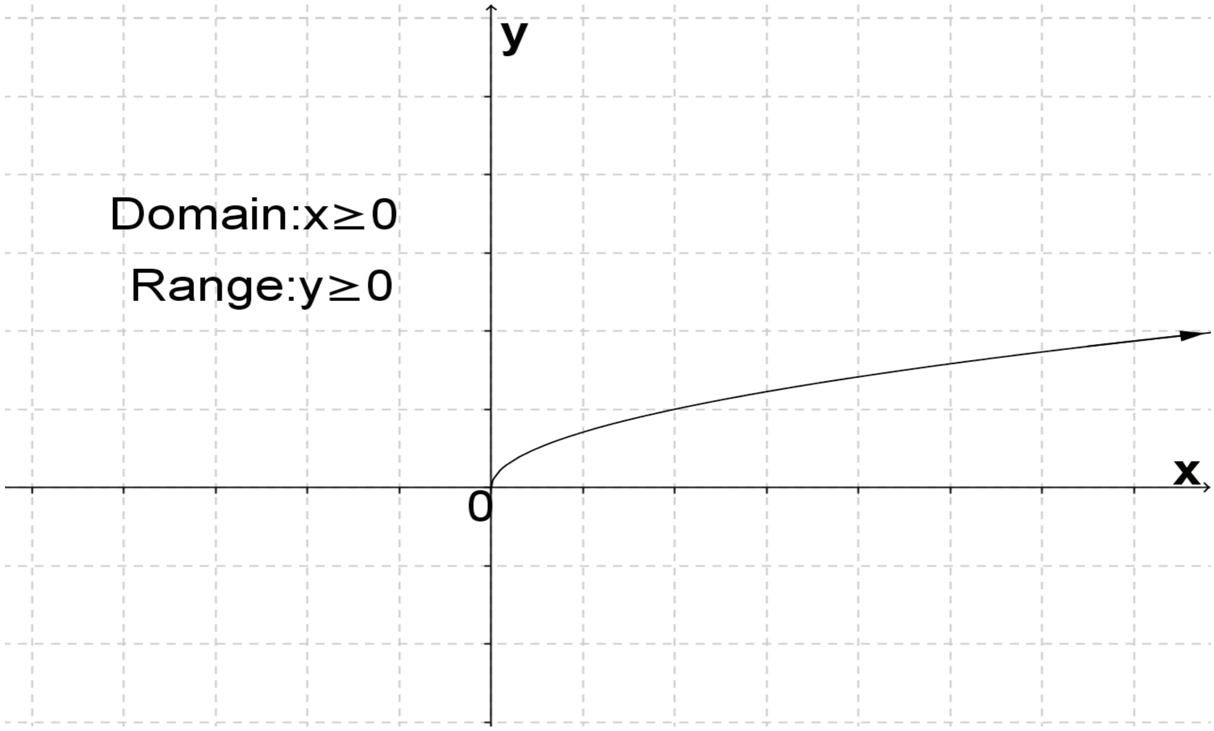
2.4	<table><tr><td>y</td><td>-3</td><td>-2</td><td>-1</td><td>0</td><td>1</td><td>2</td></tr><tr><td>x</td><td>$\frac{1}{8}$</td><td>$\frac{1}{4}$</td><td>$\frac{1}{2}$</td><td>1</td><td>2</td><td>4</td></tr></table> See sketch above for *h*.	✓ one mark for all *y*-values ✓ one mark for all *x*-values (2)
2.5		
2.5.1	No *x*-intercept	✓ answer (1)
2.5.2	$x = 1$	✓ answer (1)

2.6		
2.6.1	Domain: $x \in \mathbb{R}$ **OR** $-\infty < x < \infty$ **OR** $x \in (-\infty; \infty)$ Range: $y > 0$ **OR** $0 < y < \infty$ **OR** $y \in (0; \infty)$	✓ answer ✓ answer (2)
2.6.2	Domain: $x > 0$ **OR** $-\infty < x < \infty$ **OR** $x \in (-\infty; \infty)$ Range: $y \in \mathbb{R}$ **OR** $-\infty < y < \infty$ **OR** $y \in (-\infty; \infty)$	✓ answer ✓ answer (2)
2.6.3	The domain of one graph is the range of the other, and vice versa.	✓ answer (1)
2.6.4	Yes, both graphs are functions. They both pass the vertical-line test.	✓ answer ✓ reason (2)

2.6.5	$y = \log_2 x$	✓ answer (1)
2.6.6	Yes, the line with equation $y = x$ is the line of symmetry.	✓ answer ✓ reason (2)
2.6.7	Any relevant logical conjecture	(3)
PART 3		
3.1.1	$x = 2y^2$ or $y = \pm\sqrt{\frac{x}{2}}$	✓answer (1)
3.1.2	Turning point of f is (0; 0) and the turning point of the inverse is (0; 0)	✓ answer ✓ answer (2)
3.1.3	✓ for f ✓ the inverse g  	(2)
3.1.4	The inverse of f is not a function; it fails the vertical-line test.	✓answer ✓ reason (2)
3.1.5	$f(x) = 2x^2$, domain: $x \geq 0$ OR $x \in [0; \infty)$ $f(x) = 2x^2$, domain: $x \leq 0$ OR $x \in (-\infty; 0]$	✓one mark for each domain (2)
3.1.6	a) If the domain of f is $x \leq 0$, then the range of the inverse will be $y \leq 0$ b) If the domain of f is $x \geq 0$, then the range of the inverse will be $y \geq 0$	✓ for $x \leq 0$ and $y \leq 0$ ✓ for $x \geq 0$ and $y \geq 0$ (2)

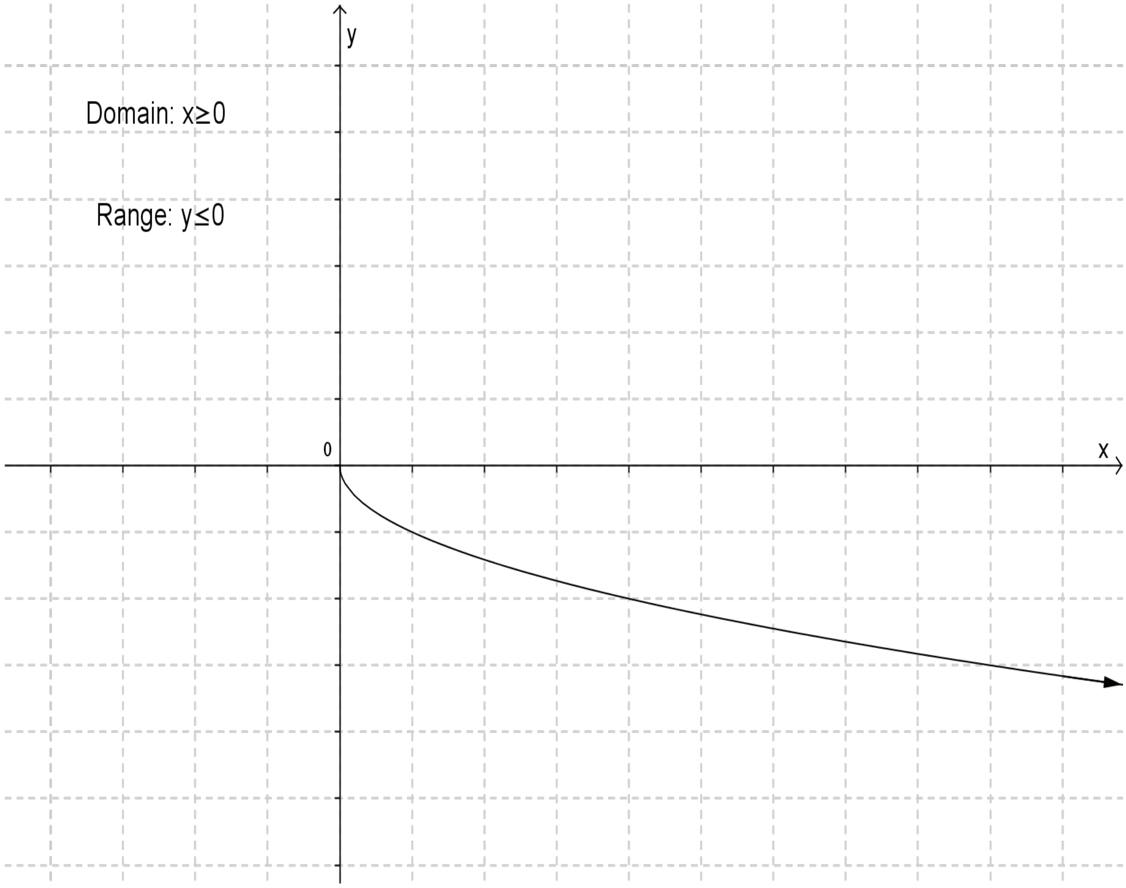
3.1.7

✓ domain
✓ correct shape



(2)

✓ domain
✓ correct shape



(2)

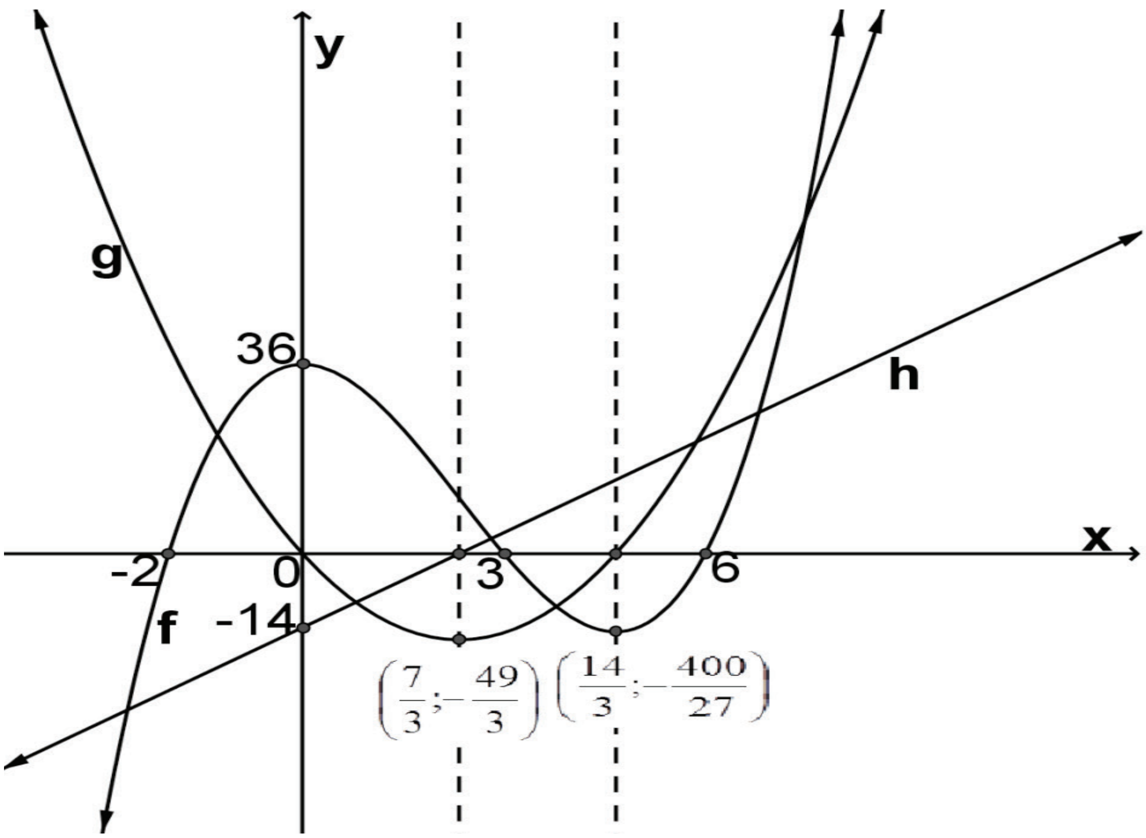
NB: The notation $f^{-1}(x) = \cdots$ is used only for one-to-one relations and may not be used for inverses of many-to-one relations as their inverses are not functions.

3. INVESTIGATION 2 **TOTAL: 50**

MEMORANDUM: APPLICATIONS OF DIFFERENTIAL CALCULUS

CASE 1	
1.$f(x) = x^3 - 7x^2 + 36$	
1.1 y-intercept = 36 for x-intercepts: $(x+2)(x^2-9x+18)=0$ $(x+2)(x-3)(x-6)=0$ $x=-2$ or $x=3$ or $x=6$ ∴ coordinates of the x-intercepts are $(-2;0)$, $(3;0)$ and $(6;0)$ For the turning points: $f'(x)=3x^2-14x$ $3x^2-14x=0$ $x(3x-14)=0$ $x=0$ or $x=\frac{14}{3}$ $f(0)=36$ TP $(0;36)$ maximum $f\left(\frac{14}{3}\right)=\left(\frac{14}{3}\right)^3-7\left(\frac{14}{3}\right)^2+36$ $=-14\frac{22}{27}$ $\left(\frac{14}{3};-14\frac{22}{27}\right)$ minimum	Marks are only awarded on the graph.
1.2 $f'(x)=3x^2-14x$ $g(x)=3x^2-14x$	1 mark for the equation $g(x)=3x^2-14$✓
1.3 $g(x)=3x^2-14x$ y-intercept = 0 For the x-intercepts: $3x^2-14x=0$ $x(3x-14)=0$ $x=0$ or $x=\frac{14}{3}$ ∴ coordinates of the x-intercepts are $(0;0)$ and $\left(\frac{14}{3};0\right)$ For the turning point: $g'(x)=6x-14$	Marks are only awarded on the graph.

$$6x-14=0$$ $$x=\frac{14}{6}$$ $$x=\frac{7}{3}$$ **OR** $$x=\frac{-b}{2a}$$ $$x=\frac{-(-14)}{2(3)}$$ $$x=\frac{14}{6}$$ $$x=\frac{7}{3}$$ $$g\left(\frac{7}{3}\right)=3\left(\frac{7}{3}\right)^2-14\left(\frac{7}{3}\right)$$ $$=\frac{-49}{3}$$ TP $\left(2\frac{1}{3};-16\frac{1}{3}\right)$	
1.4 $$g(x)=3x^2-14x$$ $$g'(x)=6x-14$$ $$h(x)=6x-14$$ y-intercept $=-14$ For the x-intercepts: $$6x-14=0$$ $$x=\frac{7}{3}$$	1 mark for the equation: $h(x)=6x-14$✓



All values are only marked from the graphs.

For *f*

x-intercepts: $x = -2;\ \ x = 3$ or $x = 6$ (1 mark for each *x*-intercept)

y-intercept: $y = 36$ (1 mark)

Turning point (0; 36) (1 mark)

Turning point $\left(\frac{14}{3}, -14\frac{22}{27}\right)$ (1 mark for each coordinate)

Shape (1 mark)

(8)

For *g*

x-intercepts: $x = 0\ or\ x = \frac{14}{3}$ (1 mark for each intercept)

Turning point $\left(\frac{7}{3}, -16\frac{1}{3}\right)$ (1 mark for both coordinates)

(3)

For *h*

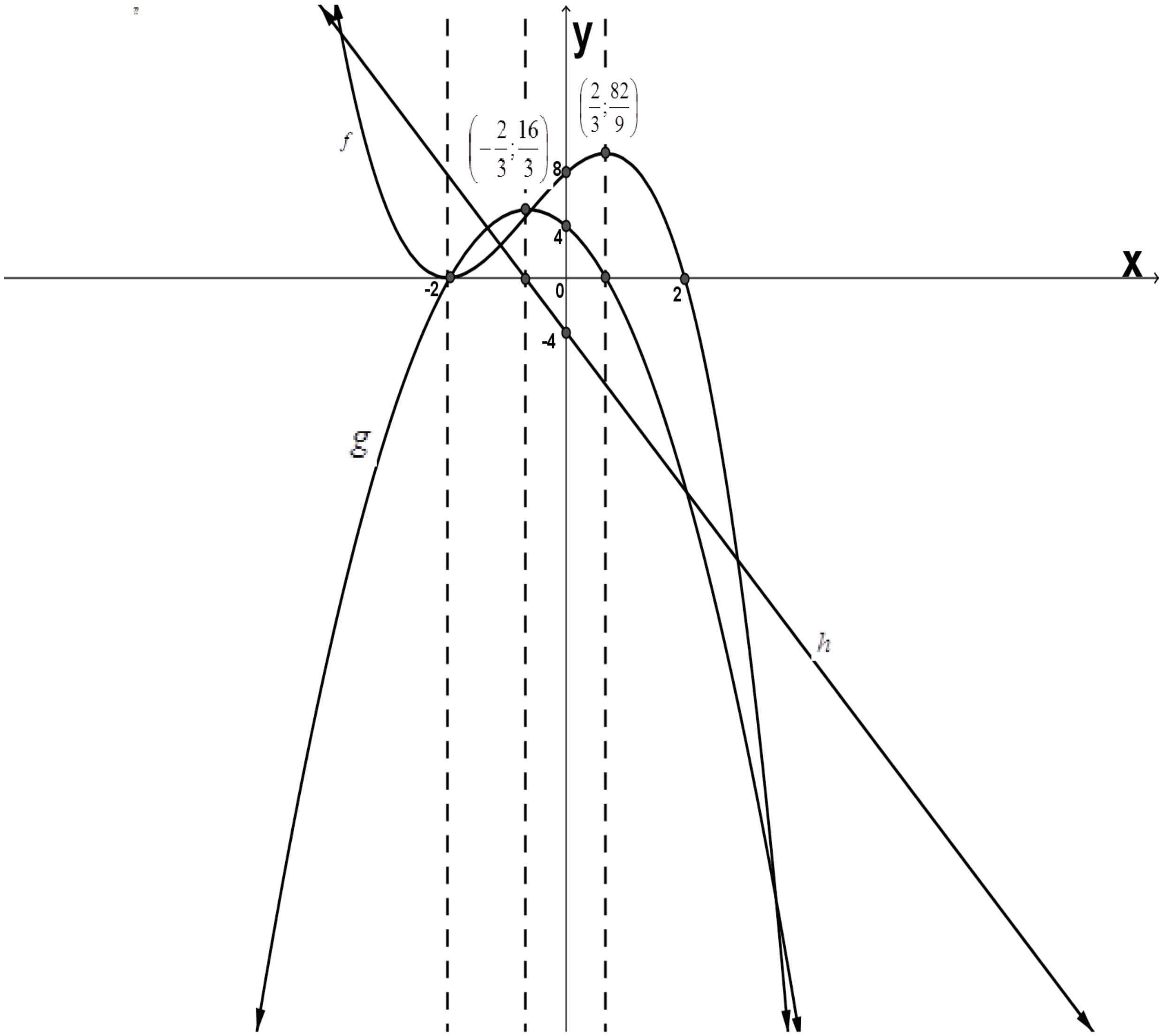
x-intercept: $x = \frac{7}{3}$

y-intercept: $y = -14$ (1 mark for each) (2)

1.5 The x-intercepts of the quadratic function and the x-coordinate of the turning point of the cubic are equal, i.e. $$x = 0 \text{ and } x = \frac{14}{3}$$	1 mark for the statement✓ (1)
1.6 $$f''(x) = 6x - 14$$ $$6x - 14 = 0$$ $$x = \frac{7}{3}$$	$f''(x) = 6x - 14$✓ $6x - 14 = 0$✓ Answer ✓ (3)
OR $$\frac{x_1 + x_2}{2} = \frac{0 + \frac{14}{3}}{2}$$ $$= \frac{7}{3}$$	✓ formula ✓ substitution ✓ answer
1.7 The axis of symmetry of g, the x-intercept of h and the point of inflection of f is $$x = \frac{7}{3}$$	✓ answer (1)
CASE 2	
2. $f(x) = -x^3 - 2x^2 + 4x + 8$	
2.1 y-intercept = 8 for x-intercepts: $$(x + 2)(x^2 - 4) = 0$$ $$(x + 2)(x - 2)(x + 2) = 0$$ $$x = -2 \text{ or } x = 2$$ ∴ coordinates of the x-intercepts are $(-2; 0)$ and $(2; 0)$ For the turning points: $$f'(x) = -3x^2 - 4x + 4$$ $$-3x^2 - 4x + 4 = 0$$ $$3x^2 + 4x - 4 = 0$$ $$(3x - 2)(x + 2) = 0$$ $$x = \frac{2}{3} \quad \text{or} \quad x = -2$$ $$f(-2) = 0$$ TP $(-2; 0)$ minimum $$f\left(\frac{2}{3}\right) = -\left(\frac{2}{3}\right)^3 - 2\left(\frac{2}{3}\right)^2 + 4\left(\frac{2}{3}\right) + 8$$ $$= 9\frac{1}{9}$$ TP $\left(\frac{2}{3}; 9\frac{1}{9}\right)$ maximum	Marks are only awarded on the graph.

2.2 $f'(x) = -3x^2 - 4x + 4$ $g(x) = -3x^2 - 4x + 4$	1 mark for equation of $g(x) = -3x^2 - 4x + 4$ ✓
2.3 y-intercept = 4 x-intercept $-3x^2 - 4x + 4 = 0$ $(3x - 2)(x + 2) = 0$ $x = -2 \quad or \quad x = \frac{2}{3}$ TP $g'(x) = -6x - 4$ $-6x - 4 = 0$ $x = \frac{-4}{6}$ $x = \frac{-2}{3}$ $g\left(\frac{-2}{3}\right) = -3\left(\frac{-2}{3}\right)^2 - 4\left(\frac{-2}{3}\right) + 4$ $g\left(\frac{-2}{3}\right) = 5\frac{1}{3}$ TP $\left(\frac{-2}{3}; \frac{16}{3}\right)$ **OR** $x = \frac{-b}{2a}$ $x = \frac{-(-4)}{2(-3)}$ $x = \frac{-2}{3}$ $g\left(\frac{-2}{3}\right) = -3\left(\frac{-2}{3}\right)^2 - 4\left(\frac{-2}{3}\right) + 4$ TP $\left(\frac{-2}{3}; \frac{16}{3}\right)$	Marks are only awarded on the graph.
2.4 $f''(x) = -6x - 4$ $h(x) = -6x - 4$ y-intercept = -4 x-intercept $-6x - 4 = 0$ $x = \frac{-2}{3}$	1 mark only for equation $h(x) = -6x - 4$✓ (1) Other marks are awarded on the graph.
2.5 The x-intercepts of the quadratic function and the x-coordinate of the turning points of the cubic are equal, i.e. $x = \frac{2}{3}$ and $x = -2$	1 mark for the statement✓ (1)

2.6 $$f''(x) = -6x - 4$$ $$-6x - 4 = 0$$ $$x = \frac{-2}{3}$$	$f''(x) = -6x - 4$ ✓ $-6x - 4 = 0$ ✓ $x = \frac{-2}{3}$ ✓ (3)
OR $$\frac{x_1 + x_2}{2} = \frac{-2 + \frac{2}{3}}{2}$$ $$= \frac{-2}{3}$$	✓ formula ✓ substitution ✓ answer
2.7 The axis of symmetry of g, the x-intercept of h and the x-coordinate of the point of inflection of f are $$x = \frac{-2}{3}$$	✓ answer (1)
3. The point of inflection of the cubic function is the same as the axis of symmetry of the graph of the first derivative and also the x-intercept of the graph of the second derivative	✓ ✓ conclusion (2)
4.	
4.1.1 for increasing: $$x < 2 \quad or \quad x > 4$$	$x < 2$ ✓ $x > 4$ ✓ (2)
4.1.2 for decreasing: $2 < x < 4$	For both values of x ✓ For correct inequality ✓ (2)
4.2 The x-values of the turning points $x = 2$ $x = 4$	$x = 2$ ✓ $x = 4$ ✓ (2)
4.3 $x = 2$ is the relative maximum since for f increasing $x < 2$ $x = 4$ is the relative minimum since for f increasing $x > 4$	$x = 2$ maximum ✓ $x = 4$ minimum ✓ (2)



All values are only marked from the graphs.

2.1 For f

Each x-intercept 1 mark $x = 2$ and $x = -2$ ✓✓ (2 marks)

y-intercept 1 mark $y = 8$ ✓

For the turning point (-2; 0) 1 mark ✓

For the turning point $(0, 67; 9{,}11)$ or $\left(\frac{2}{3}; 9\frac{1}{9}\right)$ 1 mark for x-coordinate and 1 mark for y-coordinate ✓✓(2 marks)

Shape of the graph 1 mark ✓ (7)

2.3 For g

x-intercepts: $x = -2$ ✓and $x = \frac{2}{3}$✓ (1 mark for each)

y-intercept: $y = 4$ ✓ (1 mark)

Turning point $\left(\frac{-2}{3}, 5\frac{1}{3}\right)$✓ (1 mark both coordinates) (4)

2.4 For h

x-intercept: $x = \frac{-2}{3}$ ✓ (1 mark)

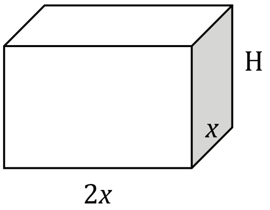
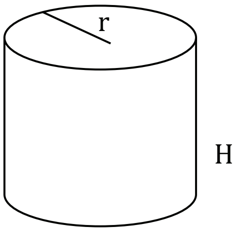
y-intercept: $y = -4$ ✓ (1 mark) (2)

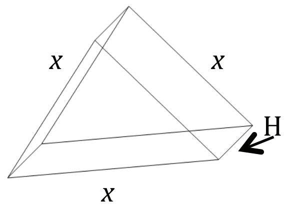
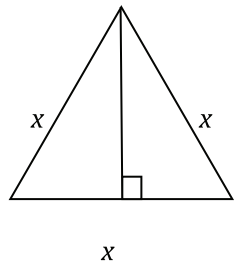
4. PROJECT **TOTAL: 50**

Applications of differential calculus

RUBRIC

CRITERIA	MAXIMUM MARK	MARK ALLOCATION		
		A	B	C
Correct mathematical formulae	3 x 3			
Correct calculations:				
Measurements of bases	4 x 3			
Height of the containers	2 x 3			
Logical reasoning and presentation	3 x 3			
Submitting on time	2			
Conclusion of the least material needed	1 x 3			
Final, further comparison	1 x 3			
Sketches	2 x 3			
TOTAL	**50**			

A	RECTANGULAR BASE  1 litre = 1 000 cm^3 Volume: $l{\times}b{\times}H$: $2x(x)(H)$ = 1 litre H = $\frac{1000}{2x^2} = \frac{500}{x^2}$ cm Surface area: $2(2x\times x) + 2(x{\times}H) + 2(2x{\times}H)$ $= 4x^2 + 2x\times\frac{500}{x^2} + 4x\times\frac{500}{x^2}$ $= 4x^2 + \frac{1000}{x} + \frac{2000}{x}$ $= 4x^2 + 3000x^{-1}$ For minimum surface area: $S'(x) = 0$ $8x - 3000x^{-2} = 0$ $x^3 = \frac{3000}{8}$ $x = \sqrt[3]{\frac{3000}{8}} = 7{,}2112 \ldots cm$ H = $\frac{500}{x^2}$ = 9,61499… cm Min surface area: $4(7{,}2112 \ldots)^2 + 3000(9{,}61499 \ldots)^{-1}$ $= 624{,}0251469\ cm^2$	1 litre = 1 000 cm^3 formula: volume H in terms of x formula: surface area Substitution of H $S'(x) = 0$ solving for $x = 7{,}2112 \ldots$ cm minimum material: substitute $x = 7{,}2112 \ldots$ to calculate H = 9,61499 … Answer
B	CIRCULAR BASE  1 litre = 1 000 cm^3 Volume: πr^2×H : = 1 litre H = $\frac{1000}{\pi r^2}$ cm^3 Surface area: $2\pi r^2 + 2\pi r{\times}H$ $= 2\pi r^2 + 2\pi r\times\frac{1000}{\pi r^2}$ $= 2\pi r^2 + 2000r^{-1}$ Min area: $S'(r) = 0$ $4\pi r - 2000r^{-2} = 0$ $r^3 = \frac{2000}{4\pi}$ $r = \sqrt[3]{\frac{2000}{4\pi}} = 5{,}419 \ldots cm$	formula: volume formula: Surface area Substitution of H $S'(r) = 0$ solving for $r = 5{,}419$ cm minimum material: substitute $r = 5{,}419$ and H = 10,8385…cm Answer

	$\text{H} = \dfrac{1000}{\pi r^2} = 10{,}8385\ldots cm^3$ Min surface area: $= 2\pi(5{,}419\ldots)^2 + 2\pi(5{,}419\ldots)\times 10{,}8385$ $= \ 553{,}58\ldots \text{cm}^2$	
C	 $h = \sqrt{x^2 - \left(\frac{x}{2}\right)^2}$ $h = \dfrac{\sqrt{4x^2 - x^2}}{2}$ $h = \dfrac{\sqrt{3}x}{2}$ 1 litre = 1 000 cm^3 Volume : $b\times h\times H$: $\frac{1}{2}x(h)(H)$ = 1 litre $\frac{1}{2}x\left(\frac{\sqrt{3}x}{2}\right)(H) = 1000$ H = $\frac{4000}{\sqrt{3}x^2}$ = cm Surface area: $2\left(\frac{1}{2}bh\right) + 3x\times H$ $= x\left(\dfrac{\sqrt{3}x}{2}\right) + 3x\times\dfrac{4000}{\sqrt{3}x^2}$ $= x^2\left(\dfrac{\sqrt{3}}{2}\right) + \dfrac{12000x^{-1}}{\sqrt{3}}$ Min area: $S'(x) = 0$ $2x\left(\frac{\sqrt{3}}{2}\right) - \frac{12000x^{-2}}{\sqrt{3}} = 0$ $x^3 = \frac{12000}{\sqrt{3}\times\sqrt{3}} =$ 4 000 $x = \sqrt[3]{4000} = $ 15,874 ... cm	1 litre = 1 000 cm^3 formula: volume H in terms of x formula: Surface area Substitution of H $S'(x) = 0$ solving for x = cm minimum material: substitute x calculate H Answer

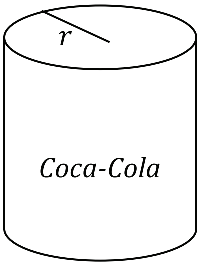
$$H = \frac{4000}{\sqrt{3}x^2} = \frac{4000}{\sqrt{3}(15,874)^2} = 9,16486\ldots\text{cm}$$

Min surface area:

$$= (15,874..)^2\left(\frac{\sqrt{3}}{2}\right) + 3(15,874\ldots.)(9,16486\ldots.)$$

$$= 654,674\ldots cm^2$$

Conclusion: The CIRCULAR BASE requires the least material to hold 1 litre of liquid.

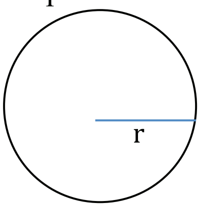


H

FURTHER COMPARISON

The Coke company uses this shape because it needs the least material in manufacturing and is therefore the most economical of the three.

The sphere: Volume : $\frac{4}{3}\pi r^3 = 1000\ cm^3$



$$r = \sqrt[3]{\left(\frac{3000}{4\pi}\right)} = 6,2035\ldots$$

Surface area $= 4\pi r^2$

$= 4\pi(6,2035\ldots)^2$

$= 483,5975862\ cm^2$

This shape requires by far the least material, but would be totally impractical. It would roll around on a flat surface. However, one could design a foot base for the can to stand on, which would then increase the cost of manufacturing.