GRADE 12

Technical Mathematics

SELF STUDY GUIDE

BOOK 2

1. DIFFERENTIAL CALCULUS

2. INTEGRATION

basic education
Department: Basic Education
REPUBLIC OF SOUTH AFRICA
Second Chance Matric Support Programme
2030 NDP
NATIONAL EDUCATION COLLABORATION TRUST
ETDP SETA
WOZA MATRICS 2021 CATCH UP
STAY SAFE
PROTECT SOUTH AFRICA
TOGETHER WE CAN BEAT THE CORONAVIRUS

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(i) INTRODUCTION

The declaration of COVID-19 as a global pandemic by the World Health Organisation led to the disruption of effective teaching and learning in many schools in South Africa. The majority of learners in various grades spent less time in class due to the phased-in approach and rotational/ alternate attendance system that was implemented by various provinces. Consequently, the majority of schools were not able to complete all the relevant content designed for specific grades in accordance with the Curriculum and Assessment Policy Statements in most subjects.

As part of mitigating against the impact of COVID-19 on the current Grade 12, the Department of Basic Education (DBE) worked in collaboration with subject specialists from various Provincial Education Departments (PEDs) developed this Self-Study Guide. The Study Guide covers those topics, skills and concepts that are located in Grade 12, that are critical to lay the foundation for Grade 12. The main aim is to close the pre-existing content gaps in order to strengthen the mastery of subject knowledge in Grade 12. More importantly, the Study Guide will engender the attitudes in the learners to learning independently while mastering the core cross-cutting concepts.

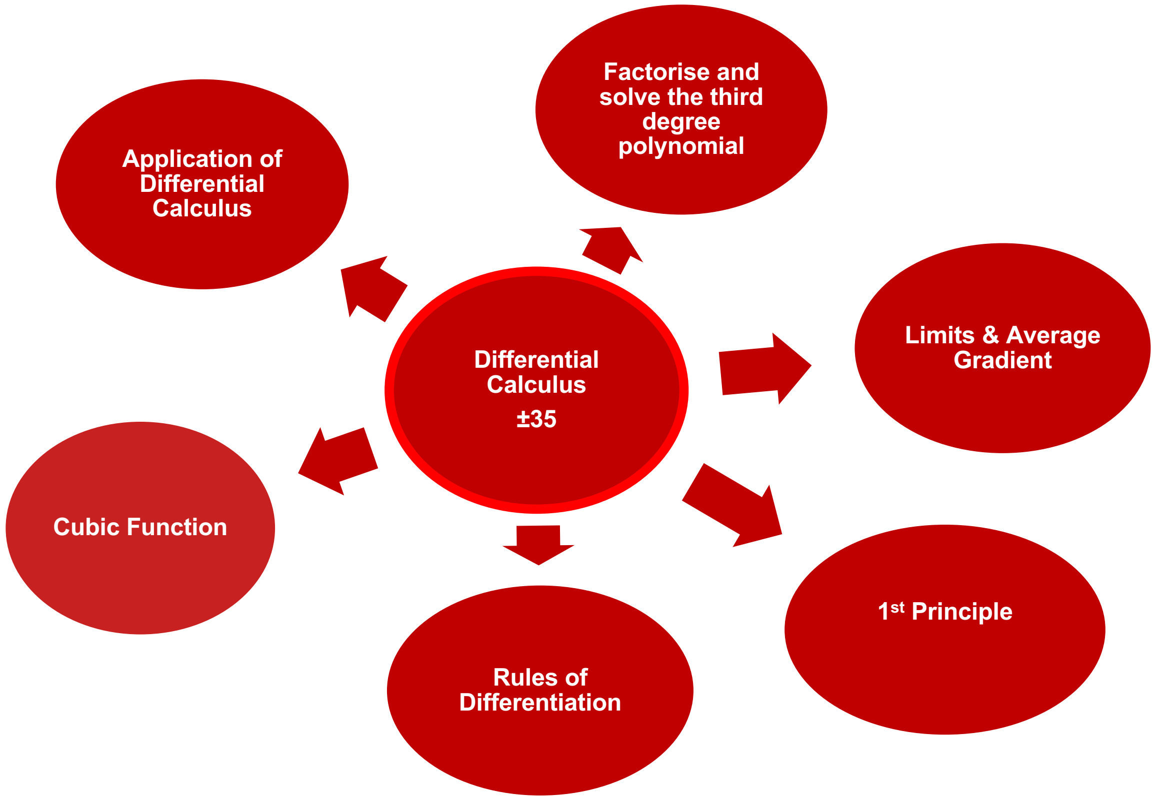
(ii) HOW TO USE THIS SELF STUDY GUIDE?

- This study guide covers two topics, namely Differential Calculus and Integration.
- In the 2021, there are three Technical Mathematics Booklets. This one is Booklet 2. Booklet 1 covers Algebra as well as Functions and Graphs while Booklet 3 covers Trigonometry and Euclidean Geometry.
- For each topic, sub-topics are listed followed by the weighting of the topic in the paper where it belongs. This booklet covers the two topics mentioned which belong to Technical Mathematics Paper 1
- Definitions of concepts are provided for your understanding
- Concepts are explained first so that you understand what action is expected when approaching problems in that particular concept.
- Worked examples are done for you to follow the steps that you must follow to solve the problem.
- Exercises are also provided so that you have enough practice.
- Selected Exercises have their solutions provided for easy referral/ checking your correctness.
- More Exam type questions are provided.

1 DIFFERENTIAL CALCULUS ($\pm$35 Marks)

- Learners will be able to:
  - Have an intuitive understanding of the concept of a limit.
  - Differentiate specified functions from first principles.
  - Use specified rule of differentiation.
  - Determine the equation of tangents to graphs.
  - Sketch graphs of cubic functions.
  - Solve practical problems involving optimisation and rate of change (including calculus of motion).
- Calculus is an interesting branch of mathematics that deals with the rate of change.
- Calculus is useful in finding the slope of any curve at a particular point, if it exists.

TOPIC MIND MAP



Study Skills

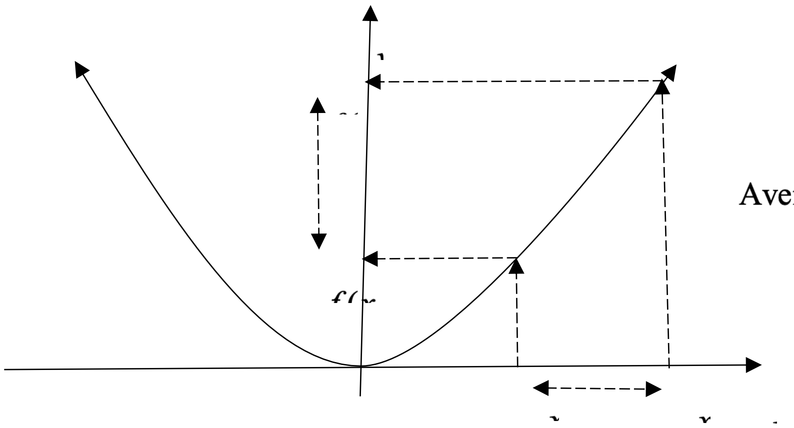
			How do I learn this?	Formula relevant to this section
Breakdown of topic	**1**	**Finding the Average gradient**	Practice by substituting into the formula.	$m_{\text{ave}} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$
	2	**Finding the derivative by using first principles**	Practice various examples of the following functions 1 $f(x) = ax$ 2 $f(x) = k$ 3 $f(x) = ax + k$ Substitute into the formula. Be very careful with the use of brackets and the correct use if the distributive property of multiplication.	$f^{/}(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$
	3	**Rules of differentiation**	1 Make sure you understand the rule: $y = x^n$ $\dfrac{dy}{dx} = nx^{n-1}$ 2 You may not differentiate when there are brackets- first multiply it out. 3 You may not differentiate when *x* is the denominator- write it as $\dfrac{1}{x} = x^{-1}$ 4 You may not differentiate when there is a surd- write it as $\sqrt{x} = x^{\frac{1}{2}}$ 5 Know the rules of exponents and simplification of expressions. 6 Always leave your answers with positive exponents 7 Study this by practicing all the sums you can find. The more you do, the better you will become.	Different notations: $\dfrac{dy}{dx}$ if $y = ...$ is given $D_x[....]$ $f^{/}(x)$ if $f(x) = ...$is given
	4	**Finding the equation of a tangent to a**	• Differentiate the function given	$y = mx + c$

		curve or function	• Then substitute by the given x to get the gradient. • Substitute the given x into the original equation to get the y-coordinate. • Use the point $(x; y)$ and the gradient to determine the equation of the tangent, i.e. substitute into the formula: $y - y_1 = m(x - x_1)$	
	5	**The Cubic Function**	1. First you must sketch about ten of these functions to really get a feel of how this works. 2. You must be able to plot giving attention to x- intercepts, y-intercepts, and stationary points. 3. After sketching you can find the equation of the cubic function. 4. There are 3 different scenarios. Make sure you are familiar with these. 5. Once you can sketch and find the equation you can move on to deductions of graphs. 6. You must know all the previous theory of graphs such as domain, range etc.	When given 3 x-intercepts and another point, then use: $y = a(x - x_1)(x - x_2)(x - x_3)$ If the graph has a turning point on one of the x-intercepts, use the equation: $y = a(x - x_1)(x - x_2)^2$ If the turning points are known, substitute into $f^{/}(x) = 0$
	6	**Calculus Application**	You need to practice all the different sums you can find from previous examination papers. You will apply all your knowledge from previous grades, especially formulas for perimeter, area, surface areas, volume, etc.	Perimeter of Rectangle= $2(l \times b)$ Area of rectangle = $l \times b$ Perimeter of square= 4s Area of square = s^2 Circumference of circle = $2\pi r$ Area of circle = πr^2 Area of a right- angled triangle= $\frac{1}{2}.b.h$

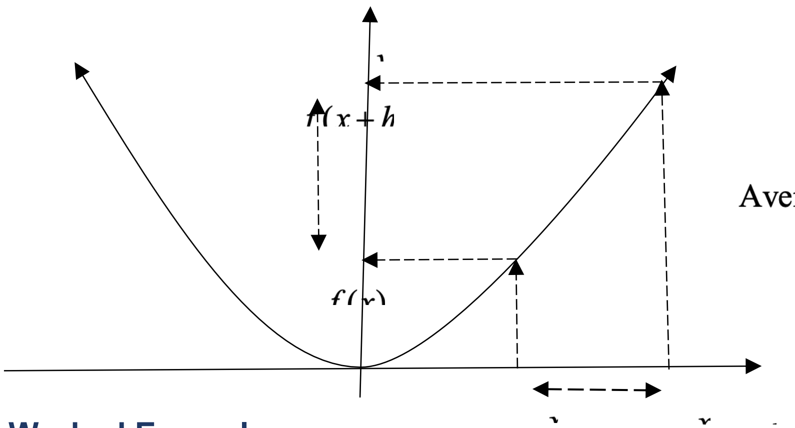
AVERAGE GRADIENT/AVERAGE RATE OF CHANGE

- The **average gradient** between any two points on a curve is the gradient of the straight line passing through the two points.
- The average gradient on the curve between two points is given by:

$m = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$ **OR** $m = \dfrac{f(x+h) - f(x)}{h}$: $h = x_2 - x_1$



$$\text{Average Gradient} = \frac{f(x_1) - f(x_2}{x_1 - x_2}$$



$$\text{Average Gradient} = \frac{f(x+h) - f(x}{h}$$

Worked Examples

Determine the average gradient of $f(x) = 3x^2$ between the two points where $x = -1$ and $x = 5$ using the formula:

(1) $m = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

(2) $m = \dfrac{f(x+h) - f(x)}{h}$

Solutions

(1)

$$
\begin{aligned}
m &= \frac{f(x_2)-f(x_1)}{x_2-x_1} \\
&= \frac{f(5)-f(-1)}{x_2-x_1} \\
&= \frac{3(5)^2-3(-1)^2}{5-(-1)} \\
&= 12
\end{aligned}
$$

(2)

$$
\begin{aligned}
m &= \frac{f(x+h)-f(x)}{h} \\
&= \frac{3(x+h)^2-3x^2}{h} \\
&= \frac{3(x^2+2xh+h^2)-3x^2}{h} \\
&= \frac{3x^2+6xh+3h^2-3x^2}{h} \\
&= \frac{6xh+3h^2}{h} \\
&= \frac{6(-1)(6)+3(6)^2}{6} \\
&= 12
\end{aligned}
$$

Substituting $f(x+h)$ and $f(x)$

$$
\begin{aligned}
h &= x_2 - x_1 \\
&= 5-(-1) \\
&= 6
\end{aligned}
$$

Practice Questions

1 Given the function $f(x) = 2x - 3$, calculate the average the average gradient between $x = 1$ and $x = 0$ (2)

2 Consider the function $f(x) = x^2 - 1$

2.1 Draw a sketch of the function f labelling the intercepts and the turning points. (2)

2.2 Determine the average gradient of f between the points where $x = -3$ and $x = 1$ (2)

3 A tennis ball is thrown into the air and has a height of $h(t) = 12 - 12t - 5t^2$ after t seconds. Find the average rate of change of the height between $t = 0$ and $t = 1$ second. (2)

4 The profit made when a company sells its produce can be described by function $p(x) = 700 - \frac{1}{5}x^2$

4.1 Determine the profit if 50 products are sold. (2)

4.2 Determine the profit if 100 products are sold. (2)

4.3 What will be the difference in profit when 50 products and 100 products are sold? (2)

4.4 Determine the average change in profit when 50 products and when 100 products are sold. (2)

5 Determine the average gradient of $f(x) = 1 - x^2$ between the points where $x = -1$ and $x = a$ (3)

Calculate the value(s) of a.

LIMITS

- When finding the limit, we look at what happens to a function value of a curve (y-value) as we get closer and closer to a specific value (x-value) of the curve.
- When writing a limit, we use the notation $\lim_{x \to c} f(x)$
  - The abbreviation ‘ lim ’ tells us that we are finding the limit.
  - The notation ‘ $x \to c$ ’ tells us which specific value the x-value is approaching.
  - $f(x)$ represents the function with which we are working
- The limit is a useful tool in calculus.
- In order to find that the limit if exists, one needs to simply substitute the *x- value* into the function *x* approaches.
- In case of complicated problems one needs to first simplify the expression problems using factorisation and other techniques before finding the limit

Worked Examples

Evaluate the following limits

(1) $\lim_{x \to 3} 3x + 2$ **(2)** $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$ **(3)** $\lim_{h \to 0} \frac{(3+h)^2 - 3^2}{h}$

Solutions

(1) $\lim_{x\to 3} 3x + 2 = 8$

(2) $\lim_{x\to 2} \frac{x^2-4}{x-2} \longrightarrow$ You cannot substitute x=2 as this would cause 0 in the denominator

$= \lim_{x\to 2} \frac{(x+2)(x-2)}{x-2} \longrightarrow$ Factorised

$= \lim_{x\to 2}(x+2)$

$= 4$

(3) $\lim_{h\to 0} \frac{(3+h)^2-3^2}{h}$

$= \lim_{h\to 0} \frac{9+6h+h^2-9}{h}$

$= \lim_{h\to 0} \frac{6h+h^2}{h}$

$= \lim_{h\to 0} \frac{h(6+h)}{h}$

$= \lim_{h\to 0}(6+h)$

$= 6$

Practice Questions

1 Determine the following limits, if they exist:

1.1 $\lim_{x \to 2} 4$ (1)

1.2 $\lim_{x \to -2} 2x$ (1)

1.3 $\lim_{x \to -3} (x^2 - 2x)$ (1)

1.4 $\lim_{x \to 2} \frac{2x^2 - 8}{x^2 - 2x}$ (3)

1.5 $\lim_{x \to 2} \frac{x^3 + 1}{x + 1}$ (2)

2 Explain why the limit of $\frac{1}{x}$ when x approaches zero does not exist. (1)

FIRST PRINCIPLES

- Assume that a straight line is drawn between two points on a curve, BUT these two points are brought closer and closer together, until their distance between the two points tends towards zero. (i.e. the two points "merge" into one point)
- If $h =$ distance between the two point, then as the distance tend towards zero, we write it as $h \to 0$.
- We can be able to find the gradient at any point of a curve called **gradient at a point/instantaneous gradient**.
- We will find the limit as h tends to 0. i.e. $\lim_{h \to 0}$
- Gradient at a point = the Derivative
- The derivative is indicated by $f'(x)$
- $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ this formula is known as **The First Principles**.

Worked Examples

Determine the derivative of the following using **FIRST PRINCIPLES**

(1) $f(x) = 4$ **(2)** $f(x) = 3x$ **(3)** $f(x) = 2x - 1$

(4) $f(x) = 7 - 5x$

Solutions

(1) $f(x) = 4$

$$f(x+h) = 4$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{4-4}{h}$$

$$= \lim_{h \to 0} \frac{0}{h}$$

$$= 0$$

(2) $f(x) = 3x$

$$f(x) = 3(x+h)$$

$$= 3x + xh$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{3x + 3h - 3x}{h}$$

$$= \lim_{h \to 0} \frac{3h}{h}$$

$$= \lim_{h \to 0} 3$$

$$= 3$$

(3) $f(x) = 2x - 1$

$$f(x+h) = 2(x+h) - 1$$

$$= 2x + 2h - 1$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{2x + 2h - 1 - 2x + 1}{h}$$

$$= \lim_{h \to 0} \frac{2h}{h}$$

$$= \lim_{h \to 0} 2$$

$$= 2$$

(4) $f(x) = 7 - 5x$

$$f(x+h) = 7 - 5(x+h)$$
$$= 7 - 5x - 5h$$

$$f^{/}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{7 - 5x - 5h - 7 - (5x)}{h}$$
$$= \lim_{h \to 0} \frac{7 - 5x - 5h - 7 + 5x}{h}$$
$$= \lim_{h \to 0} \frac{5h}{h}$$
$$= \lim_{h \to 0} 5$$
$$= 5$$

Practice Questions

1 Given the function $f(x) = 5x$, determine the following:

1.1 $f(2)$ (1)

1.2 $f(b)$ (1)

1.3 $f(x + h)$ (1)

2 If $f(x) = 3 - 2x$, then determine the following:

2.1. $f(-2)$ (2)

2.2. $f(a)$ (1)

2.3. $f(2 + h)$ (2)

2.4. $2f(x + h)$ (2)

Practice Questions

1. Determine $f^{/}(x)$ for the **FIRST PRINCIPLES**:

1.1 $f(x) = 3$ (2)

1.2 $f(x) = 4x$ (3)

1.3 $f(x) = 3x - 2$ (3)

1.4 $f(x) = 2 - 5x$ (3)

1.5 $f(x) = 2x + \pi$ (3)

1.6 $f(x) = \frac{1}{3}x + 5$ (3)

1.7 $f(x) = -3 - 2x$ (3)

2. Determine $f^{/}(x)$ from the **FIRST PRINCIPLES** if $f(x) = 0{,}5x$ (3)

RULES OF DIFFERENTIATION

Prior Knowledge

- **Before you use differentiation, you might need to simplify or change the format of the expressions:**
  - **Expand brackets** e.g. expand $(3x + 2)(x - 5) = 3x^2 - 13x - 10$ because you have no rule for differentiating a product in Grade 12. So you need separate terms before you can differentiate.
  - Rewrite terms which are square **roots**, cube roots or other roots as **exponentials**.

  Example: $\sqrt{x} = x^{\frac{1}{2}}$

- **Laws of Exponents**:

  Negative exponent : $a^{-m} = \frac{1}{a^m}$

  Fractional Exponent : $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

  Multiplying powers with same bases : $a^m \times a^n = a^{m+n}$

Practice Questions

1. Simplify the following expressions:

1.1 $(x-1)(x+2)$ (1)

1.2 $\left(x+\dfrac{1}{x}\right)^2$ (1)

1.3 $\left(4x^2\right)^3$ (1)

1.4 $\dfrac{8x^3-27}{2x-3}$ (1)

1.5 $\dfrac{7x^4-2x}{4x}$ (2)

2. Write the following without roots:

2.1 $\sqrt{x}$ (1)

2.2 $3\sqrt{x}$ (1)

2.3 $\sqrt[3]{x}$ (1)

2.4 $\sqrt[5]{x^{-2}}$ (1)

2.5 $-2\sqrt{x}+\dfrac{3}{\sqrt[3]{x^2}}$ (2)

- **DIFFERENTIATION** means finding the derivative of a function $f(x)$ with respect to x.
- The **DERIVATIVE** of a function gives the **GRADIENT** (or rate of change) of that function at any point of the curve.
  - If $f(x) = x^n$ then $f'(x) = nx^{n-1}$ (Rule in words: Multiply the coefficient of x by the exponent and then subtract 1 form the power)
  - If $f(x) = k$ (k is a constant) then $f'(x) = 0$
- **NOTATION**: $\left[\dfrac{d}{dx} \quad \dfrac{dy}{dx} \quad f'(x) \quad D_x\right]$ all means the same thing

Worked Examples

(1) $y = 2x$

(2) $f(x) = 7$

(3) $f(x) = \dfrac{x^2 - 9}{x + 3}$

(4) $\dfrac{d}{dx}\left[(x + 2\pi)(x^2 - 4)\right]$

(5) $f(x) = \dfrac{2x^3 + 5x - 4\sqrt{x}}{2x}$

(6) $D_x\left[\left(3 + \sqrt{x}\right)^2\right]$

Solutions

(1) $y = 2x$

$$\frac{dy}{dx} = 2$$

(2) $f(x) = 7$

$$f'(x) = 0$$

(3) $f(x) = \dfrac{x^2 - 9}{x + 3}$

$$f(x) = \frac{(x-3)(x+3)}{x+3} \quad \text{factorisation}$$

$$f(x) = x - 3$$

$$f'(x) = 1$$

(4) $\dfrac{d}{dx}\left[(x + 2\pi)(x^2 - 4)\right]$

$$= \frac{d}{dx}\left[x(x^2 - 4) + 2\pi(x^2 - 4)\right] \quad \text{Removing brackets}$$

$$= \frac{d}{dx}\left[x^3 - 4x + 2\pi x - 8\pi\right]$$

$$= 3x^2 - 4 + 4\pi x$$

(5) $f(x) = \dfrac{2x^3 + 5x - 4\sqrt{x}}{2x}$

$$f(x) = \frac{2x^3}{2x} + \frac{5x}{2x} + \frac{4x^{\frac{1}{2}}}{2x} \quad \text{Exponential form}$$

$$f(x) = x^2 + \frac{5}{2} + 2x^{-\frac{1}{2}}$$

$$f'(x) = 2x - x^{-\frac{3}{2}}$$

(6) $D_x\left[\left(3 + \sqrt{x}\right)^2\right]$

$$= D_x\left[(3 + \sqrt{x})(3 + \sqrt{x})\right] \quad \text{Simplification}$$

$$= D_x\left[3(3 + \sqrt{x}) + \sqrt{x}(3 + \sqrt{x})\right] \quad \text{Removing brackets}$$

$$= D_x\left[9 + 3\sqrt{x} + 3\sqrt{x} + x\right] \quad \text{Adding like terms}$$

$$= D_x\left[9 + 6\sqrt{x} + x\right]$$

$$= D_x\left[9 + 6x^{\frac{1}{2}} + x\right]$$

$$= 3x^{-\frac{1}{2}} + 1$$

Practice Questions

1 Determine:

1.1 $f^{/}(x)$ if $f(x) = 6x$ (1)

1.2 $f^{/}(x)$ if $f(x) = 6x + 5$ (1)

1.3 if $f(x) = 2x^3 - 1$ (1)

1.4 $D_x\left[x^2 + \dfrac{1}{x^2}\right]$ (2)

1.5 $\dfrac{dy}{dx}$ if $y = (2x + 3)^2$ (2)

1.6 $D_x\left[(x - 3)(3x + 1)\right]$ (2)

2 Determine:

2.1 $D_x\left[4x^2 + \dfrac{2}{x^2}\right]$ (3)

2.2 $\dfrac{dy}{dx}$ if $y = \dfrac{3}{2x} - \dfrac{x^2}{2}$ (3)

3 Determine $f^{/}(x)$ if $f(x) = 2\sqrt{x^3} + \pi x$ (3)

4 $\dfrac{dv}{d.}$ if $v = -5t^2 + 2t + \dfrac{3}{t}$ (4)

5 $\dfrac{dy}{dx}$ if $xy + 3x^2y = 3x + 1$ (4)

Solving equations in the third-degree polynomials:

$ax^3 + bx^2 + cx + d = 0$

Worked Examples

- Work through this example:
- Factorise and solve for *x*: $x^3 - x^2 - 5x = 3$
  - Get $ax^3 + bx^2 + cx + d = 0$ i.e. $x^3 - x^2 - 5x - 3 = 0$ (Standard form)
  - Use the **Factor and Remainder theorem** to find one factor.
  - Use trial and error

  This step can also be calculated on a calculator – see below.

> The factor theorem states:
>
> **If $f(k) = 0$, then $x - k$ is factor of $f(x)$**

- So if $f(x) = x^3 - x^2 - 5x - 3$, we want to find an *x*-value that makes $f(x)$
- $f(x)$ has a constant value of –3.
- If this expression can be factorised, then at least one of its factors will use a factor of –3 in it.
- The factors of –3 are: –3; –1; 1; 3

By *trial and error*, test these factors to find which value of x will result to $f(x) = 0$

If $x = -3$, then $f(-3) = (-3)^3 - (-3)^2 - 5(-3) - 3 = -24 \neq 0$

If $x = -1$, then $f(-1) = (-1)^3 - (-1)^2 - 5(-1) - 3 = 0$

$\therefore x + 1$ is a factor of $f(x)$

We will use *x* + 1 to find the other factors.

Divide $x^3 - x^2 - 5x - 3$ by $x + 1$ to find the other factors.

You can use the algebraic method, long division or synthetic division at this point.

Method I: Using algebra

$x^3 - x^2 - 5x - 3 = (x + 1)(x^2 + px - 3)$

<u>Check this</u>: First terms give x^3, last terms give –3 when multiplying with the factor

We don't know the middle terms, so we have used *px* in the second bracket.

<u>To calculate the value of *p*:</u>

The x^2 term in the expression has a coefficient of –1 $x^3 - x^2 - 5x - 3$

So the x^2 part of the factorised expression must make – x^2

$x(px) + 1(x^2) = px^2 + x^2$ $\quad (x+1)(x^2 + px - 3)$

$px^2 + x^2 = -x^2$

$px^2 = -2x^2$

$\therefore p = -2$

$f(x) = x^3 - x^2 - 5x - 3 = (x+1)(x^2 - 2x - 3)$

Method II: Long division [Divide, Multiply, Subtract, Bring down]

	$x^2 - 2x - 3$	
$x + 1$	$x^3 - x^2 - 5x - 3$	divide: $x^3 \div x$
		answer x^2 on top
	$x^3 + x^2$	multiply: $x^2(x+1)$
	$-2x^2 - 5x$	subtract & bring down
		divide: $-2x^2 \div x$
		answer $-2x$ on top
	$-2x^2 - 2x$	multiply: $-2x(x+1)$
	$-3x - 3$	subtract & bring down
	$-3x - 3$	divide: $-3x \div x$
		answer -3 on top
		multiply: $-3(x+1)$
	0	Subtract

$\therefore$ $x^3 - x^2 - 5x - 3 = (x + 1)(x^2 - 2x - 3)$

- Factorise the answer further by factorising the trinomial.

  $f(x) = x^3 - x^2 - 5x - 3 = (x+1)(x^2 - 2x - 3) = (x+1)(x-3)(x+1)$

- Determine the three solutions.

  $(x+1)(x-3)(x+1) = 0$

  $\therefore x = -1$ or $x = 3$ or $x = -1$

  Then $(x + 1) = 0$ or $(x - 3) = 0$ or $(x + 1) = 0$

  $x = -1$ or $x = 3$ or $x = -1$

  These are the x-intercepts of a cubic graph with the equation:

  $f(x) = x^3 - x^2 - 5x - 3$

Practice Questions

1 Given: $f(x)=x^3-2x^2+1$

1.1 Show that $(x-1)$ is a factor (2)

1.2 Factorise f fully. (3)

2 Show that $(x+2)$ is a factor of f if $f(x)=x^3+2x^2-3x-6$ and hence, find the other factors. (3)

3 Given: $f(x)=2x^3-3x^2+5x-14$

3.1 Show that $(x-2)$ is a factor of f (2)

3.2 Factorise f fully. (1)

4 The function $f(x)=-x^3-x^2+8x+12$

4.1 Show that $(x-3)$ is a factor of f. (2)

4.2 Factorise f fully (1)

4.3 Hence, find the values of x if $f(x)=0$ (3)

SKETCHING OF CUBIC FUNCTION

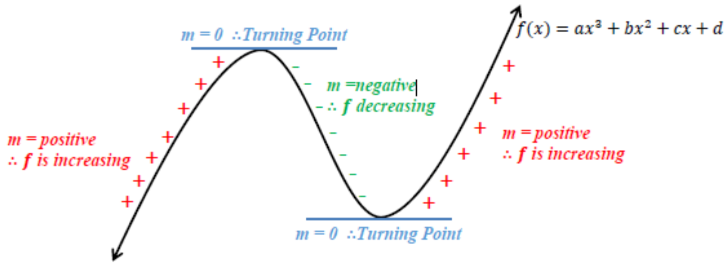
Rules of sketching the graphs of Cubic functions:

Intercepts with the Axes

- To determine the y –intercept, let $x = 0$ and solve for y
- To determine the x –intercept, let $y = 0$ and solve for x. You may be required to use Factor theorem/synthetic method/calculator or long division method to solve for x.

Stationary Points (also called Turning points or Critical points)

- When we determine $f'(x)$ we are dealing with the gradient of f which can be increasing,

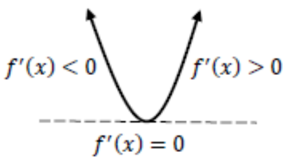


decreasing or equal to zero.

$a > 0$  1st turning point  2nd turning point

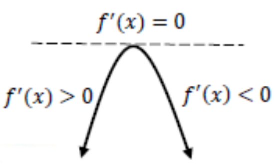
$a < 0$  1st turning point  2nd turning point

- Minimum turning points



At a minimum turning point the sign of the gradient changes from negative to positive

- Maximum turning points



At a maximum turning point the sign of the gradient changes from positive to negative

- Determine the derivative, f'(x), equate it to zero and solve for x
- Substitute the x-values of the stationary points into the original equation to obtain the corresponding y-values.
- If the function has two stationary points, establish whether they are maximum or minimum turning points by referring to the shape.

Rules of finding the equation of a given graph:

- If the 3 x-intercepts of the graph are known and another point is given, start with the equation

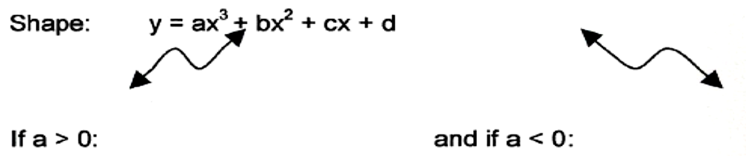
$$y=a(x-x_1)(x-x_2)(x-x_3)$$

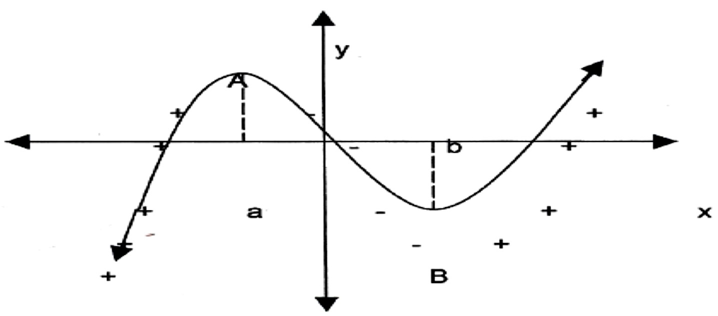
- If the graph has a turning point on one of the x-intercepts, use the equation

$$y=a(x-x_1)(x-x_2)^2$$

- If the turning points are known, substitute into $f'(x)=0$

Shapes of the cubic graphs





Equation of the tangent to the graph:

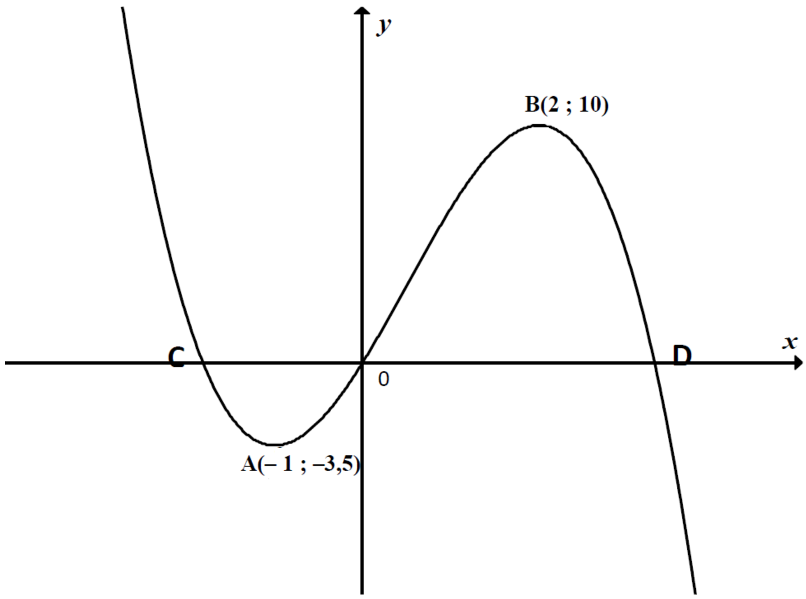
- Point (*x*; *y*)
- Gradient at x being given
- $y = mx + c$
- Determine the gradient/derivative $f^{/}(x) = 0$
- Substitute *x* into $f^{/}(x)$ to get the gradient value;
- Determine the point (*x*; *y*) on the graph (substitute x at a point of tangency into original equation) to get the corresponding *y*-value at a point of tangency (i.e. the point where the tangent touches the graph);
- Now substitute gradient-value and a point into y = mx +c to get equation of the tangent

Worked Examples

1 Sketch the graph of function $f(x) = x^3 - 8x^2 + 5x + 14$

2 Determine the equation of a tangent to a curve $y = x^2 - 6x + 9$ at a point where $x = 1$

3 The graph of $h(x) = -x^3 + ax^2 + bx$ is shown below. $\text{A}(-1;-3{,}5)$ and $\text{B}(2;10)$ are the turning points of h. The graph passes through the origin and further cuts the $x-\text{axis}$ at C and D



3.1 Show that $a = \dfrac{3}{2}$ and $b = 6$

3.2 Calculate the average gradient of A and B

3.3 Determine the equation of the tangent of h at $x = -2$

Solutions

1 Sketch the graph of function $f(x) = x^3 - 8x^2 + 5x + 14$

Stationary points: $f^{/}(x) = 0$

$$f^{/}(x) = 3x^2 - 16x + 5$$
$$3x^2 - 16x + 5 = 0$$
$$(3x-1)(x-5) = 0$$
$$\left.x = \frac{1}{3} \text{ or } x = 5\right\} \boxed{x \text{ - values of stationary points}}$$

Substituting *x*-values of stationary point in $f(x)$ to get corresponding y-values

$$f\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)^3 - 8\left(\frac{1}{3}\right)^2 + 5\left(\frac{1}{3}\right) + 14$$
$$= \frac{400}{27}$$
$$= 14{,}8$$

$$f(5) = (5)^3 - 8(5)^2 + 5(5) + 14$$
$$= -36$$

Finding *x*-intercept

$$f(x) = x^3 - 8x^2 + 5x + 14$$
$$f(-1) = (-1)^3 - 8(-1)^2 + 5(-1) + 14$$
$$= 0$$
$$\therefore x+1 \text{ is a factor of } f(x)$$

$$f(x) = x^3 - 8x^2 + 5x + 14$$
$$f(x) = (x+1)(ax^2 + bx + c)$$

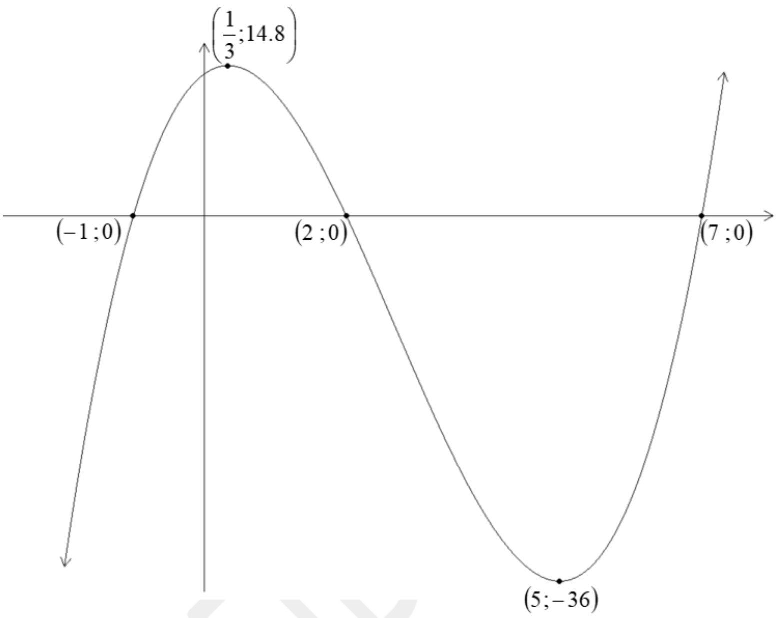
$ax^3 = x^3$	$ax^2 + bx^2 = -8x^2$
$a = 1$	$a + b = -8$
$c = 14$	$b = -8 - 1$
	$b = -9$

$$\therefore 0 = (x+1)(x^2 - 9x + 14)$$
$$0 = (x+1)(x-7)(x-2)$$
$$x = -1 \text{ or } x = 7 \text{ or } x = 2$$

or

$$\begin{array}{cccc|c} & -1 & 9 & -14 & \\ 1 & -8 & 5 & 14 & 1 \\ \hline 1 & -9 & 14 & 0 & \end{array}$$

$$\therefore (x+1)(x^2 - 9x + 14) = 0$$
$$\therefore (x+1)(x-7)(x-2) = 0$$
$$\therefore x = -1 \text{ or } x = 7 \text{ or } x = 2$$



2 Determine the equation of a tangent to a curve $y = x^2 - 6x + 9$ at a point where $x = 1$

Point of contact; $x = 1$

$y = (1)^2 - 6(1) + 9$

$y = 4$

$(1;4)$

Co-ordinates of point of contact are $(1;4)$

$f^{/}(x) = 2x^2 - 6$

$f^{/}(1) = 2(1) - 6$

$= -4$

$\therefore m_{\tan} = -4$

Substituting the co-ordinates of point of contact $(1;4)$ into $y = mx + c$

$y = mx + c$

$4 = -4(1) + c$

$c = 8$

$\therefore y = -4x + 8$

3.1 $h'(x) = -3x^2 + 2a + b$

$h'(-1) = -3(-1)^2 + 2a(-1) + b$

$0 = -3 - 2a + b$

$2a - b = -3 \quad .... \quad (i)$

$h'(2) = -3(2)^2 + 2a(2) + b$

$0 = -12 + 4a + b$

$4a + b = 12 \quad .... \quad (ii)$

Adding (i) and (ii)

$2a + 4a - b + b = -3 + 12$

$6a = 9$

$\therefore a = \frac{3}{2}$

Substituting $a = \frac{3}{2}$ into (i)

$\therefore 2\left(\frac{3}{2}\right) - b = -3$

$3 - b = -3$

$b = 6$

3.2 Average gradient $= \frac{10 - (-3,5)}{2 - (-1)}$

$= \frac{13,5}{3}$

$= \frac{9}{2}$

3.3 $h(x) = x^3 + \frac{3}{2}x^2 + 6x$

$\therefore h'(x) = -3x^2 + 2x + 6$

$h'(-2) = -3(-2-)^2 + 3(-2) + 6$

$h'(-2) = -12$

Point of Contact $(-2;2)$

$y - y_1 = m(x - x_1)$

$y - 2 = -12(x - (-2))$

$y = 12x - 22$

Practice Questions

1 $f(x) = x^3 - 9x^2 + 24x - 16$

1.1 Write down the y-intercept of the graph of f. (1)

1.2 If $x - 1$ is a factor of f, determine the other factors and hence write down the coordinates of the $x-$ intercepts of f. (3)

1.3 Determine the turning points of the graph of f. (5)

1.4 Draw sketch graph of *f*. Clearly indicate all the intercepts with the axes and the turning points. (4)

2 Given the function $f(x) = (x + 2)(x - 2)^2$

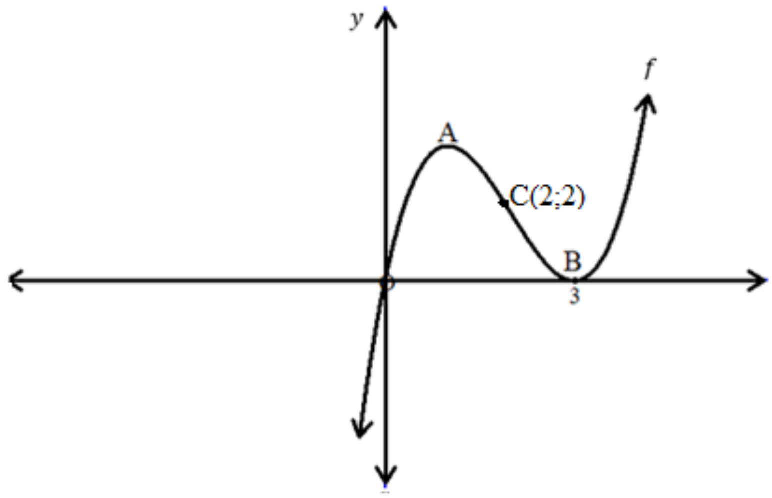
2.1 Determine the $y-$ intercepts of the graph. (2)

2.2 Write down the $x-$intercepts of the graph. (1)

2.3 Determine the stationary points of f. (5)

2.4 Draw sketch graph of *f*. Clearly indicate all the intercepts with the axes and the turning points. (3)

3 The sketch below represents the graph of f defined by $f(x) = ax^3 + bx^2 + c$ A and B are the turning points of f. B (3;0) is also the x – intercept of f. C is a point with coordinates (2;2). The graph passes through the origin.

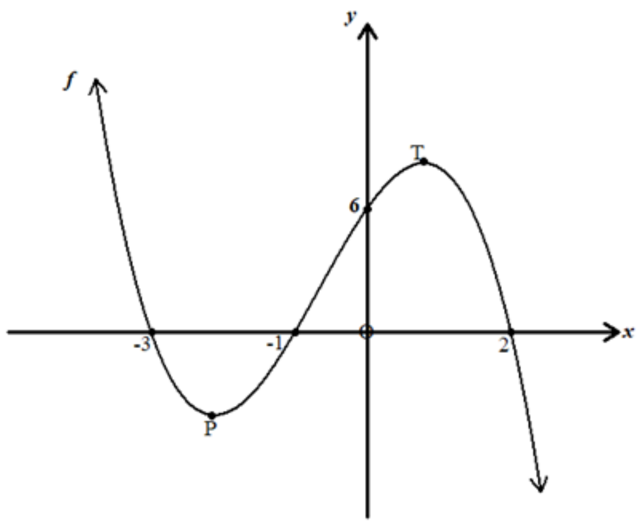


3.1 Show that the equation of f is $f(x) = x^3 - 6x^2 + 9x$ (3)

3.2 Determine the coordinates of A, the turning point of f. (5)

3.3 For which value(s) of f will $f(x) \leq 0$? (1)

4 The sketch below represents the graph of f defined by $f(x) = -x^3 + ax^2 + bx + c$. P and T are the turning points of f. The x-intercepts of the graph are – 3; – 1 and 2. The graph has a y–cut at 6.



4.1 Determine the values of a, b and c. (3)

4.2 Determine the coordinates of T and P, the turning points of the graph, if $f(x) = -(x+3)(x+1)(x-2)$. (5)

4.3 For which values of x will $f(x) \geq 0$? (2)

4.4 Determine the gradient of the tangent to the graph of f at $x = 2$ (2)

5 Determine the equation of the tangent to the graph $y = x^3 + 4x^2 + 2x - 3$ at $x = -2$ (6)

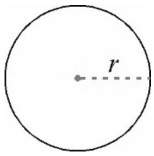
6 The gradient of a tangent to $f(x) = x^3 - 3x + 2$ at the point $(r\ ;\ t)$ is 24.
Determine the values of r and t if $r > 0$ (4)

FINDING THE MAXIMUM AND MINIMUM

- $f^{/}(x) = 0$ shows us the local maximum or minimum points. We can use this to solve an applied problem that asks for a maximum or minimum value.
- This is revision of Grade 10 work that is needed in order to help you with some Grade 12 questions about measurement, volume, maximums and minimums. You need to know these formulae and use them to solve problems.

2-D shapes	**3-D shapes** **Right prisms**	**3-D shapes** **Where the base is a polygon and the sides meet at one point, the apex.**
Area **&** **Perimeter** (The distance around the outside)	V = Area of base × ⊥ height & Surface area = the sum of the areas of the flat shapes	$V = \frac{1}{3}\text{Area of base} \times \text{height}$ $= \frac{1}{3} A \times H$ where H is the perpendicular height **&** $\text{Surface area} = \text{Area of base} + \frac{1}{2}\text{ph}$ where p is the perimeter of the base and h is the slant height

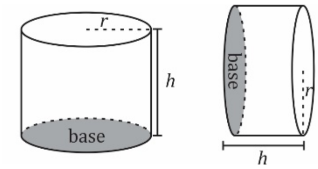
1 Circle



$A = \pi r^2$

$\text{Circumference} = 2\pi r$

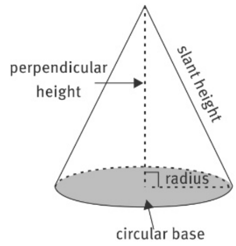
1 Right cylinders



$V = \pi r^2 h$

$\text{Surface area} = 2\pi r^2 + 2\pi rh$

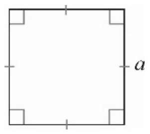
1 Cones



$V = \frac{1}{3}\pi r^2 \times \mathrm{H}$

$$\begin{aligned}\text{Surface area} &= \pi r^2 + \frac{1}{2}(2\pi r \times h)\\ &= \pi r^2 + \pi rh\end{aligned}$$

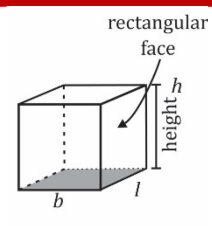
2 Square



$A = \text{length} \times \text{length} = a^2$

$\text{Perimeter} = 4a$

2 Square prism

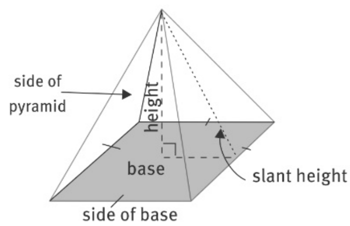


Note: $l = b = h = a$

$V = a \times a \times a = a^3$

$\text{Surface area} = 6a^2$

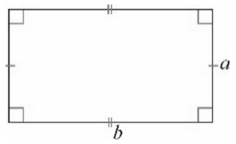
2 Square base pyramid



$V = \frac{1}{3}a^2 \times \mathrm{H}$

$$\begin{aligned}\text{SA} &= \text{A of square} + 4 \times \text{A of } \Delta\\ &= a^2 + 4\left(\frac{1}{2} \times a \times h\right)\\ &= a^2 + 2ah\end{aligned}$$

3 Rectangle

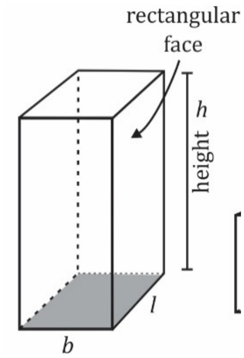


A = length × breadth

$= ab$

Perimeter $= 2a + 2b$

3 Rectangular prism



$V = l \times b \times h$

$SA = 2lb + 2lh + 2bh$

The **slant height** runs from the middle of the edge of the base to the apex.

We calculate the slant heights using the perpendicular height and the dimensions of the base by using the Theorem of Pythagoras.

1 An open box is to be made out of a rectangular pieces of card measuring 64cm by 24cm . Figure 1 show how a square of side length x cm is to be cut of each corner so that the box can be made by folding, in figure 2

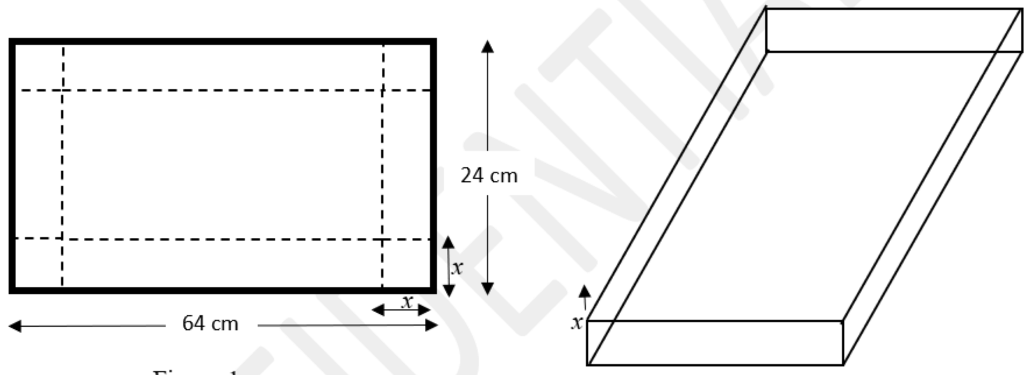


Figure 1 Figure 2

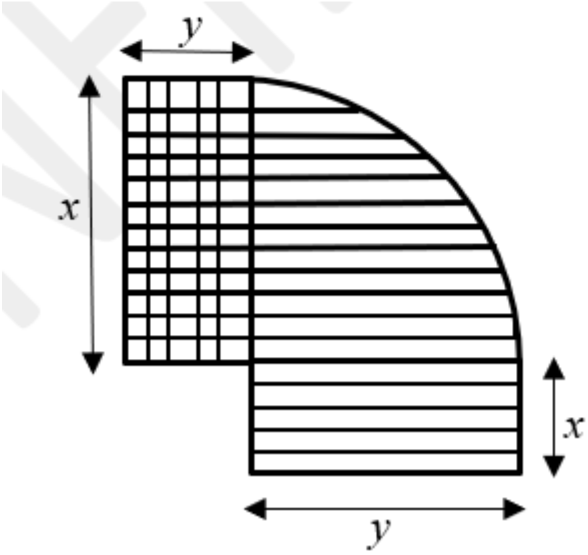
1.1 Show that the volume of the box, $V\,cm^3$, is given by:
$V = 4x^3 - 176x^2 + 1536x$

1.2 Show further that the stationary points V occur when
$3x^2 - 88x + 1536 = 0$

1.3 Determine the value of *x* for which V is stationary.

1.4 Determine to the nearest cm^3, the maximum value of V.

2 The figure below shows the design for an earring consisting of a quarter circle with two identical rectangles attached to either straight edge of the quarter circle. The quarter circle has radius *x* cm and each of the rectangles measures *x* cm by *y* cm.



The earring is assumed to have a negligible thickness and treated as a two dimensional object with area of 12,25 cm^2.

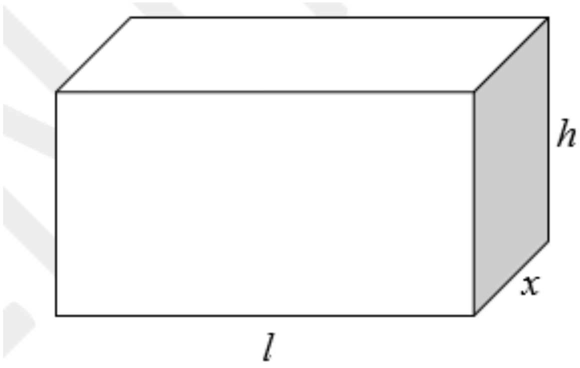
2.1 Show that $4y = \frac{49 - \pi x^2}{2x}$. Hint: Use the area of the object.

2.2 Determine the perimeter of the earring in the form of *x*.

2.3 Determine the value of *x* that makes the perimeter of the earring minimum.

2.4 Show that for the value of x calculated in (b), the corresponding y value is $y = \frac{7}{16}(4 - \pi)$

3 A rectangular box is constructed in such a way that the length (l) of the base is three times as long as its width. The material used to construct the top and the bottom of the box costs R100 per square metre. The material used to construct the sides of the box costs R50 per square metre. The box must have a volume of 9 m^3. Let the width of the box be x metres.



3.1 Determine an expression for the height (h) of the box in terms of x.

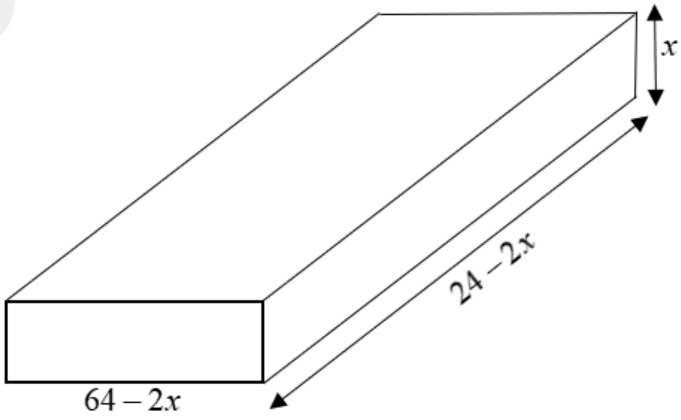
3.2 Show that the cost to construct the box can be expressed as:

$$C = \frac{1200}{x} + 600x^2$$

3.3 Calculate the width of the box (that is the value of x) if the cost is to be minimised.

Solutions

1



1.1 $V = x(64-2x)(24-2x)$

$= x\left(1536 - 128x - 48x + 4x^2\right)$

$= x\left(1536x - 176x + 4x^2\right)$

$= 1536x - 176x^2 + 4x^3$

1.2 Stationary point: $V' = 0$

$V' = 1536 - 352x + 12x^2$

$0 = 1536 - 352x + 12x^2$

$\frac{12x^2}{4} - \frac{352x}{4} + \frac{1536}{4} = 0$

$3x^2 - 88x + 384 = 0$

1.3 $3x^2 - 88x + 384 = 0$

$(3x-16)(x-24) = 0$

$x = \frac{16}{3}$ or $x = 24$

$x \neq 24$ because box is 24cm long

1.4 When $x = \frac{16}{3}$

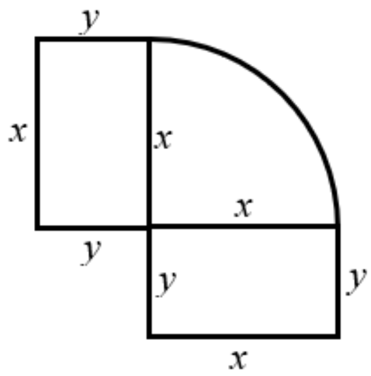
$V = \frac{16}{3}\left(64 - 2\left(\frac{16}{3}\right)\right)\left(24 - 2\left(\frac{16}{3}\right)\right)$

$= \frac{16}{3}\left(64 - \frac{32}{3}\right)\left(24 - \frac{32}{3}\right)$

$= \frac{102400}{27}$

$\approx 3792{,}59$

2



2.1 $A = 2(l \times b) + \frac{1}{4}\pi r^2$

$12{,}25 = 2(x \times y) + \frac{1}{4}\pi x^2$

$49 = 8xy + \pi x^2$

$\frac{8xy}{2x} = \frac{49 - \pi x^2}{2x}$

$4y = \frac{49}{2x} - \frac{\pi x}{2}$

2.2 $P = 2x + 4y + \frac{1}{4}(2\pi r)$

$= 2x + \frac{49 - 4\pi x}{2} + \frac{1}{4}(2\pi x)$

$= 2x + \frac{49}{2x} - \frac{4\pi x}{2} + \frac{\pi x}{2}$

$= 2x + \frac{49}{2x} - \frac{\pi x}{2} + \frac{\pi x}{2}$

$\therefore P = 2x + \frac{49}{2x}$

2.3 $P = 2x + \frac{49x^{-1}}{2}$

$\frac{dP}{dx} = 2 - \frac{49x^{-2}}{2}$

$\frac{dP}{dx} = 0$

$0 = 2 - \frac{49x^{-2}}{2}$

$0 = 2 - \frac{49}{2x^2}$

$4x^2 = 49$

$\frac{4x^2}{4} = \frac{49}{4}$

$x^2 = 12{,}25$

$\therefore x = 3{,}5$

2.4 Substitute $x = 3{,}5$ in $4y = \frac{49}{2x} - \frac{\pi x}{2}$

$4y = \frac{49}{2(3{,}5)} - \frac{\pi(3{,}5)}{2}$

$4y = 7 - \frac{7\pi}{4}$

$\frac{4y}{4} = \frac{7 - \frac{7\pi}{4}}{4}$

$y = \frac{7}{4} - \frac{7\pi}{16}$

$y = \frac{7(4 - \pi)}{16}$

$y = \frac{7}{16}(4 - \pi)$

3.1 $V = l \times b \times h$

$9 = 3x \times x \times h$

$9 = 3x^2 h$

$\therefore h = \dfrac{3}{x^2}$

3.2 $C = \left(2(3xh) + 2xh\right) \times 50 + \left(2 \times 3x^2\right) \times 100$

$= 8x\left(\dfrac{3}{x^2}\right) \times 50 + 600x^2$

$= \dfrac{1\ 200}{x^2} + 600x^2$

3.3 $C = 1\ 200x^{-1} + 600x^2$

$\dfrac{dC}{dx} = -1\ 200x^{-2} + 1\ 200x$

$-\dfrac{1\ 200}{x^2} + 1\ 200x = 0$

$1\ 200x^3 = 1\ 200$

$x^3 = 1$

$\therefore x = 1$

CALCULUS OF MOTION

- Distance: $s = f(t)$; where s is the distance covered and t represents the time taken, $t \geq 0$ [Distance is function of time]
- Speed (Velocity): $v = \dfrac{ds}{dt}$ which represents $\dfrac{\text{Change in distance}}{\text{Change in time}}$ or $f'(t)$
- Acceleration: $a = \dfrac{dv}{dt}$ which represents $\dfrac{\text{Change in distance}}{\text{Change in time}}$ or $v'(t)$

1 A ball is thrown upwards so that its height above the ground after time t is $h = 20t - 5t^2$ where h is measured in metres and t is measured in seconds.
Determine:

1.1 The equation that represents the velocity of the ball.

1.2 The time taken by the ball to reach its maximum height.

1.3 The velocity of the ball when it is 15 metres high on its way downwards.

2 A tourist travels in a car over a mountainous pass during his trip. The height above sea level of the car, after t minutes, is given as $s(t) = 5t^3 - 65t^2 + 200t + 100$ metres. The journey lasts 8 minutes.

2.1 How high is the car above sea level when it starts its journey on the mountainous pass?

2.2 Calculate the car's rate of change of height above sea level with respect to time, 4 minutes after starting the journey on the mountainous pass.

2.3 The velocity of the ball when it is 15 metres high on its way downwards.

2.4 Interpret your answer in **QUESTION** 2.2

2.5 How many minutes after the journey has started will the rate of change of height with respect to time be a minimum?

Worked Examples
Solutions

1.1 $h = 20t - 5t^2$

$v = h^{/} = 20 - 10t$

1.2 At maximum $h^{/} = 0$

$v = 20 - 10t$

$0 = 20 - 10t$

$10t = 20$

$t = 2$

The ball reaches its maximum height after 2 seconds

1.3 $h = 20t - 5t^2$

$15 = 20t - 10t^2$

$0 = -5t^2 + 20t - 15$

$0 = t^2 - 4t + 3$

$0 = (t-3)(t-1)$

$t = 3$ or $t = 1$

The ball is 15 metres high at $t = 1\text{s}$ or $t = 3\text{s}$. It is on its way downwards when $t = 3\text{s}$.

$v = 20 - 10(3)$

$= 20 - 30$

$= -10\,\text{m.s}^{-1}$

Since the ball is going downwards, its velocity is negative, and the velocity of the ball on its way down at a height of 15 metres is $-10\,\text{m.s}^{-1}$.

$s(t) = 5t^3 - 65t^2 + 200t + 100$

2.1 $t = 0$, therefore, it is $5(0)^3 - 65(0)^2 + 200(0) + 100 = 100$ metres

2.2 $s^{/}(t) = 15t^2 - 130t + 200$

$s^{/}(4) = 15(4)^2 - 130(4) + 200$

$= -80$ metres per minute

2.3 The height of the car above sea level is decreasing at 80 metres per minute and the car is travelling downwards hence it is a negative rate of change.

2.4

$$s^{/}(t) = 15t^2 - 130t + 200$$
$$v(t) = s^{/}(t)$$
$$v(t) = 15t^2 - 130t + 200$$
$$v^{/}(t) = 30t - 130$$
$$v^{/}(t) = 0$$
$$0 = 30t - 130$$
$$30t = 130$$
$$t = \frac{130}{30}$$
$$\therefore t = 4{,}33\,\text{s}$$

Practice Questions

1 The displacement of a particle, in metres, from point B, after t seconds, is given by $s(t) = 2t^2 - 25t + 300$

1.1 How far is the particle from B initially? (2)

1.2 How fast is the particle moving exactly 5 seconds? (2)

1.3 What is the minimum distance between the particle and point B? (3)

2 During an experiment the temperature T (in degrees Celsius) varies with time *t* (in hours) according to the formula: $\mathrm{T}(t) = 50 + 3t - 0{,}8t^2$

2.1 Determine an expression for the rate of change of temperature with time. (1)

2.2 During which time did the temperature start dropping? (2)

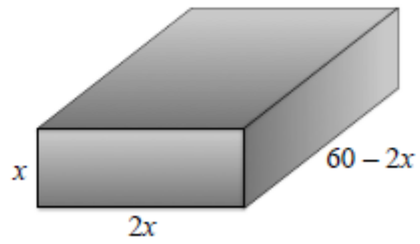
3 Air is pumped into a spherical balloon at a rate of 200cm^3/s. When the radius of the balloon is 15 cm, determine:

[HINT: $\mathrm{V} = \frac{4}{3}\pi r^3$ and Surface area, $\mathrm{S} = 4\pi r^2$

3.1 The rate at which the radius is increasing. (3)

3.2 The rate at which the surface area is increasing. (3)

4 A rectangular box has the dimensions: $x, 2x$, and $60-2x$

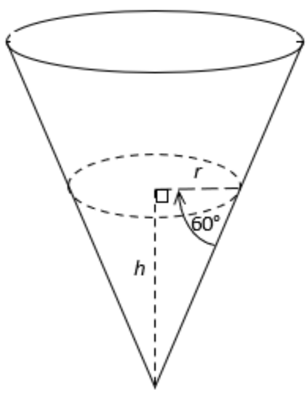


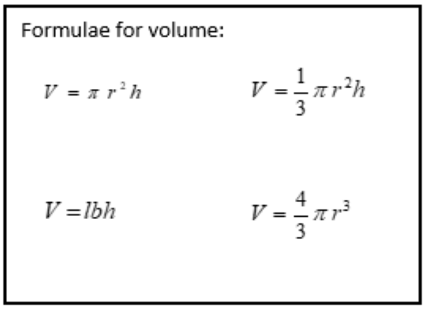
4.1 Express the volume of the rectangular box in terms of *x* (3)

4.2 Determine the dimensions of the box that would give the maximum volume (5)

4.3 Calculate the maximum volume of the rectangular box. (2)

5 A rain gauge is in the shape of a cone. Water flows into the gauge. The height of the water is *h* cm when the radius is *r* cm. The angle between the cone edge and the radius is 60°, as shown in the diagram below.

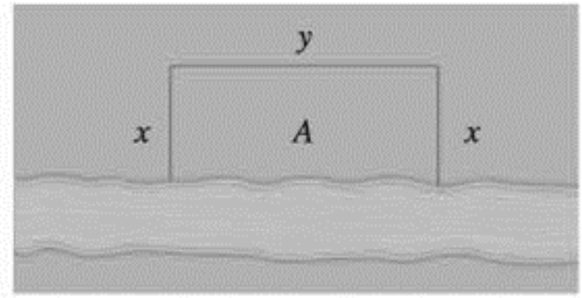




5.1 Determine *r* in terms of *h*. Leave your answer in surd form. (3)

5.2 Determine the derivative of the volume of water with respect to *h* then *h* is equal to 9 cm. (4)

6 Sipho wants to start a flower garden, which he decides to fence off in the shape of a rectangle from the rest of the garden. Michael has only 80 m of fencing, so he decides to use a wall as one border of the flower garden.



6.1 Write the y in terms of x. (2)

6.2 Calculate the width and length of the garden that corresponds to the largest possible area that Sipho can fence off. (4)

2. INTEGRATION

Fundamental rules of integration

Integrals and derivatives have numerous applications in science and engineering. For example, they are used to calculate surface areas and to formulate physical laws of electrodynamics.

- The general solution of integrals of the form $\int x^n\,dx$ where n is a constant and $n \neq -1$ is given by $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$; $C\in\mathbb{R}$
- The general solution of integrals of the form $\int ax^n\,dx$ where a and n are constants and $n \neq -1$ is given by $\int ax^n\,dx = \dfrac{ax^{n+1}}{n+1} + C$; $C\in\mathbb{R}$
- The Integral of the reciprocal of x:
  - The general solution of integrals of the form $\int \frac{1}{x}\,dx$ is given by $\int \frac{1}{x}\,dx = \ln x + C$

    $\ln x = \log_e x$ and $C\in\mathbb{R}$
  - The general solution of integrals of the form $\int \frac{a}{x}\,dx$ where a is a constant is given by

    $\int \frac{a}{x}\,dx = a\ln x + C$ and $C\in\mathbb{R}$

Function, $f(x)$	**Indefinite integral $\int f(x)\,dx$**
$f(x)=k$ where k is a constant	$\int k\,dx = kx + C$; where $C\in\mathbb{R} : f$ is Linear
$f(x)=2x$ Linear	$\int 2x\,dx = \dfrac{2x^{1+1}}{1+1} + C$ $= 2x^2 + C$; $C\in\mathbb{R}$
$f(x)=3x^2$ Parabolic function	$\int 3x^2\,dx = \dfrac{3x^{2+1}}{2+1} + C$ $= 3x^2 + C$; $C\in\mathbb{R}$
$f(x)=kx^n$ Exponential function	$\int ak^x\,dx = \dfrac{ak^x}{\ln k} + C$; $C\in\mathbb{R}$
$f(x)=x^{-1}=\dfrac{1}{x}$ Hyperbola	$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln x + C$; $C\in\mathbb{R}$
$f(x)=ax^{-1}=\dfrac{a}{x}$ Hyperbolic function	$\int \frac{a}{x}\,dx = a\ln x + C$ and $C\in\mathbb{R}$

Properties of definite integrals

Here are some simple properties of the integral that are often used in computations. Throughout take ***f*** and ***g*** as continuous functions.

1	$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
2	Constants may be factored through the integral sign: $\int_b^a kf(x)\,dx = k\int_b^a f(x)\,dx$
3	The integral of a sum (and /or difference) of integrands is the sum (and /or) difference) of the integrals: $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)\,dx + \int_a^b g(x)dx$

Worked examples

Indefinite Integrals

Determine:

1 $\displaystyle\int\left(\frac{2}{x} - 7^{x} + 3x^{2}\right)dx$

2 $\displaystyle\int\left(x(x^{2} - 2\right)dx$

3 $\displaystyle\int\left(\sqrt[4]{x} + \frac{2}{x^{2}}\right)dx$

Solutions

1	$\displaystyle\int\left(\frac{2}{x} - 7^{x} + 3x^{2}\right)dx$	
	$\displaystyle= 2\ln x - \frac{7}{\ln x} + x^{3} + C$	integrating each term
2	$\displaystyle\int\left(x(x^{2} - 2\right)dx$	
	$\displaystyle= \int\left(x^{3} - 2x\right)dx$	product
	$\displaystyle= \frac{x^{4}}{4} - x^{2} + C$	integrating each term
3	$\displaystyle\int\left(\sqrt[4]{x} + \frac{2}{x^{2}}\right)dx$	
	$\displaystyle= \int\left(x^{\frac{1}{4}} + 2x^{-2}\right)dx$	simplifying surd and fraction
	$\displaystyle= \frac{4x^{\frac{5}{4}}}{5} - 2x^{-1} + C$	integrating each term

Practice questions

Determine:

1 $\int (x+3)^2\, dx$

2 $\int \sqrt{x}\left(x + x^2\right) dx$

3 $\int \left(\dfrac{x^2 - 9}{x - 3}\right) dx$

4 $\int \left(\sqrt[3]{x^2} + \dfrac{3}{x^2}\right) dx$

5 $\int \left(\dfrac{2x}{3} - \dfrac{5}{x^4}\right) dx$

6 $\int \left(-\dfrac{6}{x} + \dfrac{1}{2}\pi p\right) dx$

7 $\int \left(5^x - m^4\right) dx$

8 $\int \left(x^{-2} + \dfrac{1}{x}\right) dx$ (3)

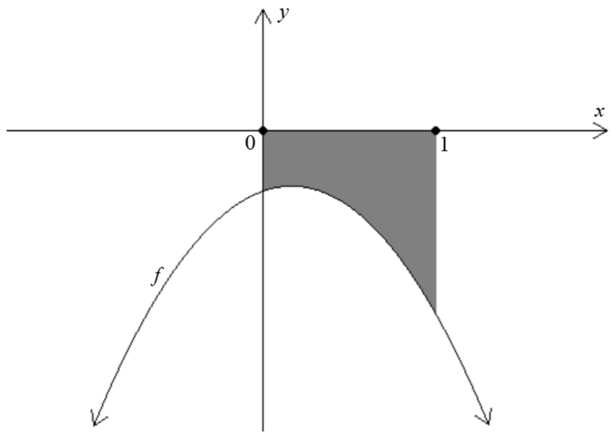
9 $\int \left(x^{\frac{1}{3}} - 5x^4\right) dx$ (2)

Worked examples

Definite Integrals

The sketch represents the bounded area of the curve of the function defined by

$f(x) = -3x^2 + x - 1$



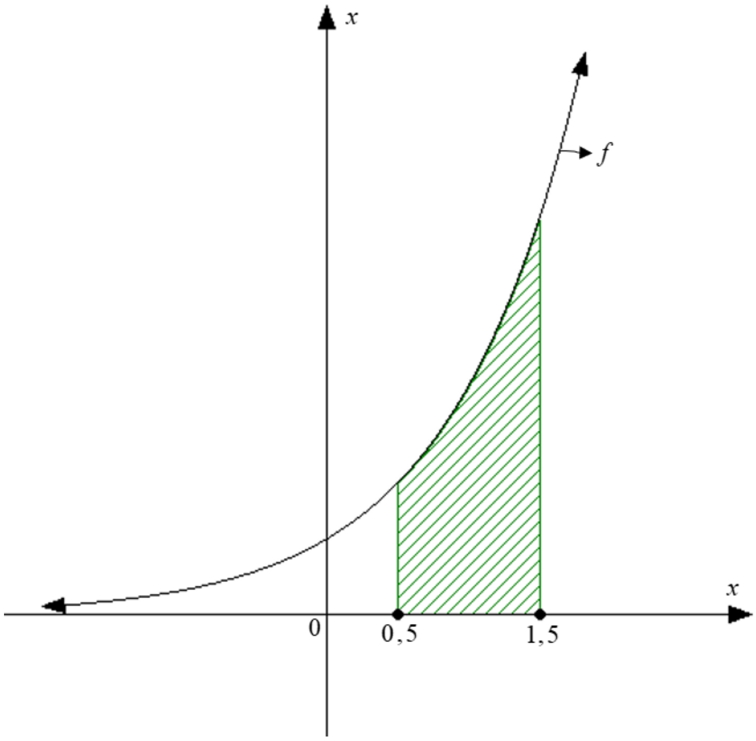
Determine the shade area bounded by the curve and the *x*-axis between the points where $x = 0$ and $x = 1$ (6)

Solutions

1	$A = \int_0^1 \left(-3x^2 + x - 1\right) dx$ $= \left(-x^3 + \frac{x^2}{2} - x\right)\Big\|_0^1$ $= \left(-(1)^3 + \frac{(1)^2}{2} - 1\right) - \left(-(0)^3 + \frac{(0)^2}{2} - 0\right)$ $= -\frac{3}{2}$ $\therefore A = \frac{3}{2} = 1{,}5$ square units **OR** $A = -\int_0^1 \left(-3x^2 + x - 1\right) dx$ $= -\left(-x^3 + \frac{x^2}{2} - x\right)\Big\|_0^1$ $= -\left(-(1)^3 + \frac{(1)^2}{2} - 1\right) - \left(-(0)^3 + \frac{(0)^2}{2} - 0\right)$ $= -\left(-\frac{3}{2}\right)$ $\therefore A = \frac{3}{2} = 1{,}5$ square units

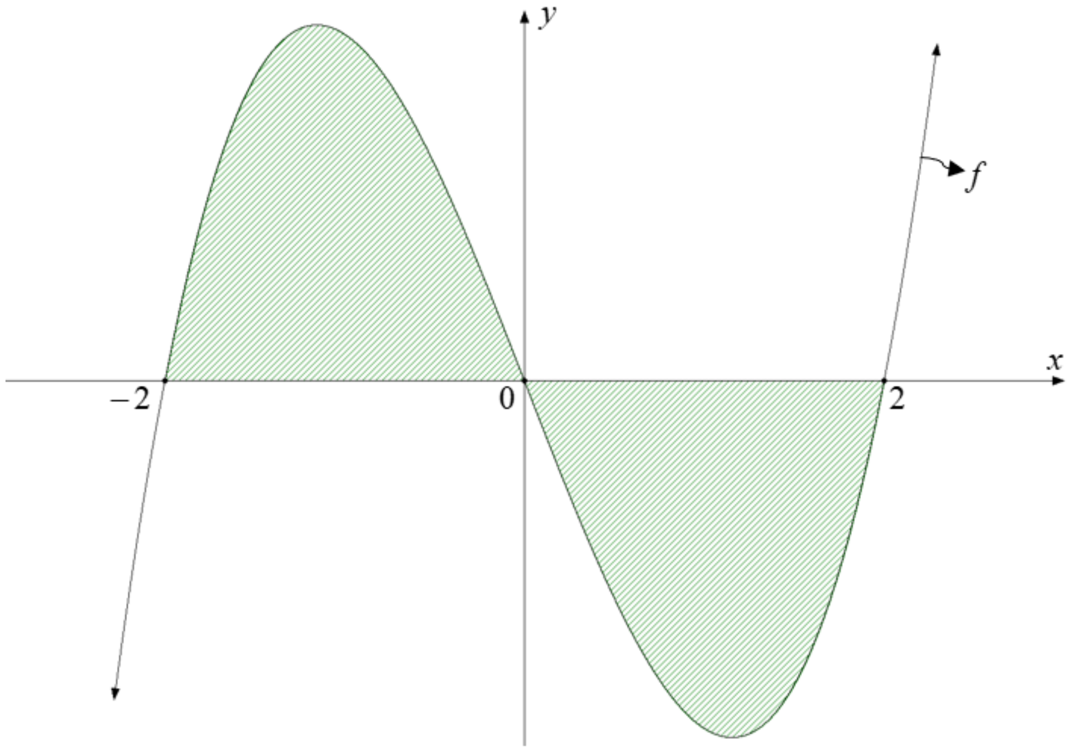
Practice questions

1 The sketch below represents the shaded bounded area of the curve of the function defined by $f(x) = 3^x$



Determine (showing ALL calculations) the shaded area bounded by the curve and the *x*-axis between the points where $x = 0,5$ and $x = 1,5$

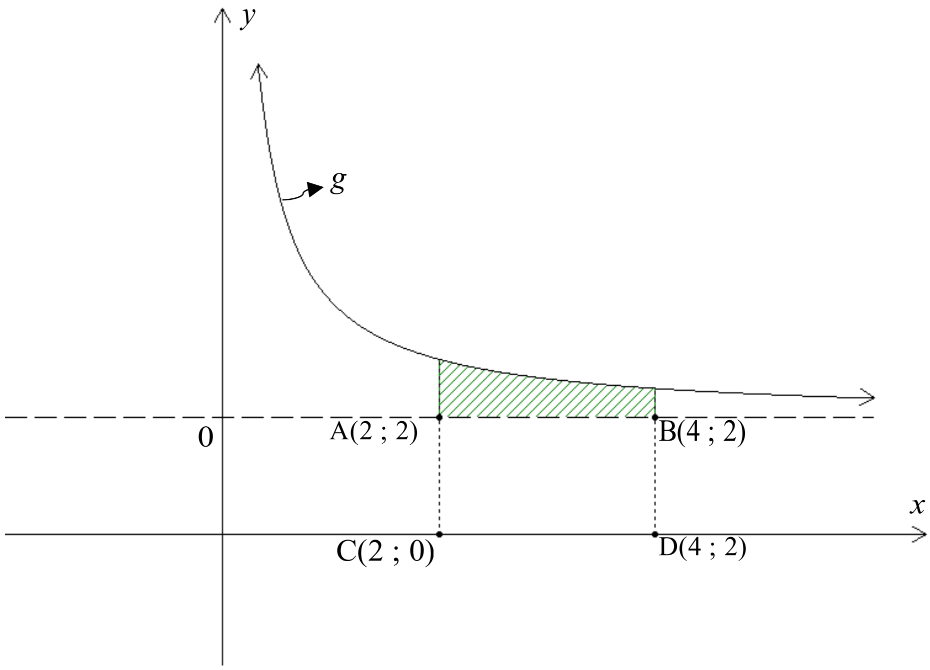
2 The sketch below represents the shaded bounded area of the curve of the function defined by $f(x) = 3x^3 - 12x$



Determine (showing ALL calculations) the shaded area bounded by the curve and the x-axis between the points where $x = -2$ and $x = 2$

1 The sketch below represents the shaded bounded area of the curve of the function defined by $g(x)=\dfrac{2}{x}+2\,;\ x>0$

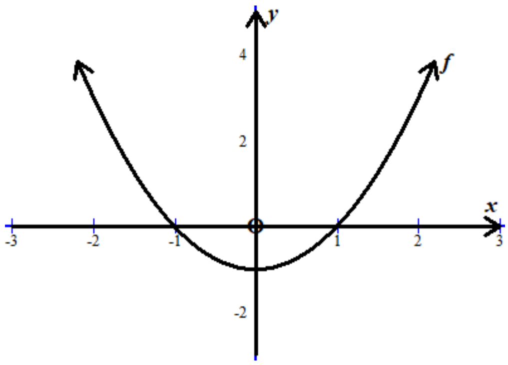
$C(2\,;0)$ and $D(4\,;2)$ are points on the *x*-axis.



Determine (showing ALL calculations) the shaded area bounded by the curve and the line $y=2$ between the points $A(2\,;2)$ and $B(4\,;2)$ (6)

Differential Calculus Marking Guidelines

Average gradient

1.1	$m=\dfrac{f(1)-f(0)}{1-0}$ $=\dfrac{[2(1)-3]-[2(0)-3]}{1}$ $=\dfrac{-1-(-3)}{1}$ $=2$	✓subst. ✓answer (2)
2.1		✓shape ✓x-and y-intercepts (2)
2.2	$m=\dfrac{f(1)-f(-3)}{1-(-3)}$ $=\dfrac{[(1^2-1)]-[(-3)^2-1]}{4}$ $=\dfrac{0-8}{4}$ $=-2$	✓subst. ✓answer (2)
3	$Aver.rate=\dfrac{f(1)-f(0)}{1-0}$ $=\dfrac{[12-12(1)-5(1)^2]-[12-12(0)-5(0)^2]}{1}$ $=\dfrac{-25-0}{1}$ $=-25$	✓subst. ✓answer (2)

4.1

$$P(50) = 900 + 0,2(50)^2$$
$$= 1400$$

✓subst.
✓answer
(2)

4.2

$$P(50) = 900 + 0,2(100)^2$$
$$= 2900$$

✓subst.
✓answer
(2)

4.3

$$Aver.change\,in\,profit = \frac{f(100) - f(50)}{100 - 50}$$
$$= \frac{2900 - 1400}{50}$$
$$= 30$$

✓subst.
✓answer
(2)

5

$$m = \frac{f(1) - f(a)}{1 - a}$$
$$= \frac{[1 - 1^2] - [1 - a^2]}{1 - a}$$
$$= \frac{-1 + a^2}{1 - a}$$
$$= \frac{\cancel{(a-1)}(a+1)}{-\cancel{(a-1)}}$$
$$= -a - 1$$

✓subst.
✓factorisation
✓answer
(3)

Limits

1.1	4	✓answer.	(1)
1.2	– 4	✓answer.	(1)
1.3	15	✓answer.	(1)
1.4	$\lim_{x\to 2}\frac{2(x^2-4)}{x(x-2)}$	✓common factor	
	$=\lim_{x\to 2}\frac{2(x-2)(x+2)}{x(x-2)}$	✓factorisation	
	$=\lim_{x\to 2}\frac{2(x+2)}{x}$		
	$=4$	✓ answer	(3)
1.5	$\lim_{x\to 1}\frac{(x+1)(x^2-x+1)}{(x+1)}$	✓factorization	
	$=\lim_{x\to 1}(x^2-x+1)$	✓ simplification	
	$=1$	✓ answer.	(2)

Function values

1.1	$f(2)=5$	✓answer.	(1)
1.2	$f(b)=5$	✓answer.	(1)
1.3	$f(x+h)=5$	✓answer.	(1)
2.1	$f(-2)=3-2(-2)$	✓subst.	
	$=7$	✓answer.	(2)
2.2	$f(a)=3-2a$	✓answer.	(1)
2.3	$f(2+h)=3-2(2+h)$	✓subst.	
	$=3-4-2h$		
	$=-1-2h$	✓answer.	(2)
2.4	$2f(x+h)=2[3-2(x+h)]$	✓subst.	
	$=6-4x-4h$	✓answer.	(2)

First Principles

1.1 $f^{/}(x)=\lim\limits_{h\to 0}\dfrac{3-3}{0}$ ✓subst.

$=0$ ✓answer. (2)

1.2 $f^{/}(x)=\lim\limits_{h\to 0}\dfrac{4(x+h)-4x}{h}$ ✓subst.

$=\lim\limits_{h\to 0}\dfrac{4x+4h-4x}{h}$ ✓simplification

$=\lim\limits_{h\to 0}\dfrac{4\cancel{h}}{\cancel{h}}$ ✓answer (3)

$=4$

1.3 $f^{/}(x)=\lim\limits_{h\to 0}\dfrac{3(x+h)-2-(3x-2)}{h}$ ✓subst.

$=\lim\limits_{h\to 0}\dfrac{3x+3h-2-3x+2}{h}$ ✓simplification

$=\lim\limits_{h\to 0}\dfrac{3\cancel{h}}{\cancel{h}}$ ✓answer. (3)

$=3$

1.4 $f^{/}(x)=\lim\limits_{h\to 0}\dfrac{2-5(x+h)-(2-5x)}{h}$ ✓subst.

$=\lim\limits_{h\to 0}\dfrac{2-5x-5h-2+5x}{h}$ ✓simplification

$=\lim\limits_{h\to 0}\dfrac{-5h}{h}$ ✓answer. (3)

$=-5$

1.5 $f^{/}(x)=\lim_{h\to 0}\frac{2(x+h)+\pi-(2x+\pi)}{h}$ ✓subst.

$=\lim_{h\to 0}\frac{2x+2h+\pi-2x-\pi}{h}$ ✓simplification

$=\lim_{h\to 0}\frac{2h}{h}$ ✓answer. (3)

$=2$

1.6 $f^{/}(x)=\lim_{h\to 0}\frac{\frac{1}{3}(x+h)+5-\left(\frac{1}{3}(x+h)+5\right)}{h}$ ✓subst.

$=\lim_{h\to 0}\frac{\frac{1}{3}x+\frac{1}{3}h+5-\frac{1}{3}x-5}{h}$ ✓simplification

$=\lim_{h\to 0}\frac{\frac{1}{3}h}{h}$

$=\frac{1}{3}$ ✓answer (3)

1.7 $f^{/}(x)=\lim_{h\to 0}\frac{-3(x+h)-(-3x)}{h}$ ✓subst.

$=\lim_{h\to 0}\frac{-3x-3h+3x}{h}$ ✓simplification

$=\lim_{h\to 0}\frac{-3h}{h}$

$=-3$ ✓answer (3)

2 $f^{/}(x)=\lim_{h\to 0}\frac{0,5(x+h)-(0,5x)}{h}$ ✓subst.

$=\lim_{h\to 0}\frac{0,5x+0,5h-0.5x}{h}$ ✓simplification

$=\lim_{h\to 0}\frac{0,5h}{h}$

$=0,5$ ✓answer. (3)

Simplification of fractions and Surds

1.1	x^2+x-2	✓answer.	(1)
1.2	$x^2+2+\frac{1}{x^2}$	✓answer.	(1)
1.3	$64x^6$	✓answer.	(1)
1.4	$\frac{(2x-3)(4x^2+6x+9)}{(2x-3)}$	✓factorisation	
	$4x^2+6x+9$	✓answer	(2)
1.5	$\frac{7x^4}{4x}-\frac{2x}{4x}$	✓simplification	
	$\frac{7}{3}x^3-\frac{1}{2}$	✓answer	(2)
2.1	$x^{\frac{1}{2}}$	✓answer.	(1)
2.2	$3x^{\frac{1}{2}}$	✓answer.	(1)
2.3	$x^{\frac{1}{3}}$	✓answer.	(1)
2.4	$x^{-\frac{2}{5}}$	✓answer.	(1)
2.5	$-2x^{\frac{1}{2}}+\frac{3}{x^{\frac{2}{3}}}$	✓simplification	(1)

Rules of differentiation

1.1	$f^{/}(x)=6$	✓answer	(1)
1.2	$f^{/}(x)=6$	✓ answer	(1)
1.3	$f^{/}(x)=6x^2$	✓ answer	(1)
1.4	$f(x)=x^2+x^{-2}$	✓simplification	
	$f^{/}(x)=2x-\frac{2}{x^3}$	✓ answer	(2)
1.5	$g(x)=(2x+3)(2x+3)$	✓simplification	
	$=4x^2+12x+9$		
	$g^{/}(x)=8x+12$	✓answer	(2)
1.6	$f(x)=3x^2-8x-3$	✓simplification	
	$f^{/}(x)=6x-8$	✓answer.	(2)
2.1	$D_x[4x^2+2x^{-2}]$	✓simplification	
	$=8x-\frac{4}{x^3}$	✓$8x$	
		✓ $-\frac{4}{x^3}$	(2)
2.2	$y=\frac{3x^{-1}}{2}+\frac{1}{2}x^2$	✓simplification	
	$\frac{dy}{dx}=\frac{3}{2x^2}+x$	✓ $\frac{3}{2x^2}$	
		✓x	(3)
3	$f(x)=2x^{\frac{3}{2}}+\pi x$	✓simplification	
	$f'(x)=3x^{\frac{1}{2}}+\pi$	✓ $3x^{\frac{1}{2}}$	
		✓ $+\pi$	
			(3)
4	$h(t)=-5t^2+2t+3t^{-1}$	✓simplification	
	$h'(t)=-10t+2-\frac{3}{t^2}$	✓ $-10t$	
		✓ $+2$	
		✓ $-\frac{3}{t^2}$	(4)

5 $y(x+3x^2)=3x+1$ ✓common factor

$y=\dfrac{3x+1}{3x^2+x}$ ✓simplification

$y=\dfrac{(3x+1)}{x(3x+1)}$

$y=\dfrac{1}{x}$ ✓ x^{-1}

$y=x^{-1}$

$\dfrac{dy}{dx}=x^{-2}=\dfrac{1}{x^2}$ ✓ $\dfrac{1}{x^2}$

(4)

Polynomials

1.1 $f(1)=1^3-2(1)+1$ ✓subst.

$=0$

$\therefore x-1$ is a factor ✓answer. (2)

1.2 $(x-1)(x^2-x-1)$ ✓✓second factor (2)

2 $f(-2)=(-2)^3+2(-2)^2-3(-2)-6$ ✓subst

$=-8+8+6-6$

$=0$ ✓answer =0

$\therefore x+2$ is a factor

$(x+2)(x^2-3)$ ✓other factor (3)

3.1 $f(2)=2(2)^3-3(2)^2+5(2)-14$ ✓subst.

$=16-12+10-14$

$=0$ ✓answer. (2)

$\therefore x-2$ is a factor

3.2 $(x-2)(2x^2+x+7)$ ✓answer. (1)

4.1 $f(3)=-(3)^3-(3)^2+8(3)+12$ ✓subst.

$=-27-9+24+12$

$=0$ ✓answer

$x-3$ is a factor

(2)

4.2 $(x-3)(-x^2+4x-4)$ ✓answer (1)

4.3 $(x-3)(-x^2+4x-4)=0$ ✓equating to zero

$(x-3)(x^2-4x+4)=0$ ✓factorising

$(x-3)(x-2)^2=0$

$x=3\,/\,x=2$ ✓both answers. (3)

Cubic graphs

1.1 $y=-16$ ✓answer. (1)

1.2 $(x-1)(x^2-8x+16)=0$ ✓factors

✓factorising

$(x-1)(x-4)^2=0$

$x=1\,/\,x=4$ ✓both answers (3)

1.3 $f'(x)=3x^2-18x+24$ ✓derivative

$3x^2-18x+24=0$ ✓equating to zero

$x^2-6x+8=0$

$(x-4)(x-2)=0$ ✓factorising

$x=4\,/\,x=2$ ✓both answers

$y=(4)^3-9(4)^2+24(4)-16$

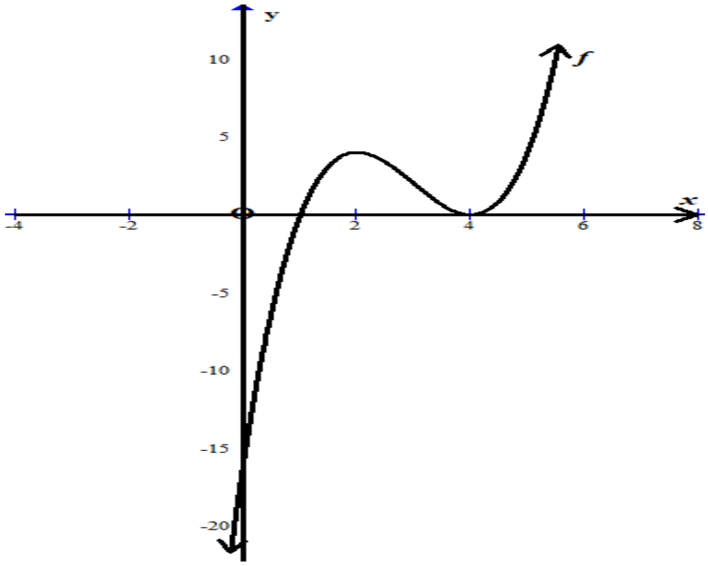
$y=0$

$y=(2)^3-9(2)^2+24(2)-16$

$y=4$

$\therefore tpts\ (4;0)$ and $(2;4)$ ✓both turning points (5)

1.4



✓shape

✓both turning points

✓x-and y-intercepts.

(3)

2.1 $y=(x-2)(x^2-4x+4)$

$y=x^3-2x^2-4x+8$

$y-\text{int}=8$

✓factors
✓answer

(2)

2.2 $x=-2\,/\,x=2$

✓both answers. (1)

2.3 $f^{/}(x)=3x^2-4x-4$

$3x^2-4x-4=0$

$$x=\frac{4\pm\sqrt{16-4(3)(-4)}}{6}$$

$$x=2\,/\,x=-\frac{2}{3}$$

$y=(2)^3-2(2)^2-4(2)+8=0$

$$y=\left(-\frac{2}{3}\right)^3-2\left(-\frac{2}{3}\right)^2-4\left(-\frac{2}{3}\right)+8=9,5$$

$$\therefore (2,0) \quad \text{and} \quad \left(-\frac{2}{3};9,5\right)$$

✓derivative

✓equating to zero

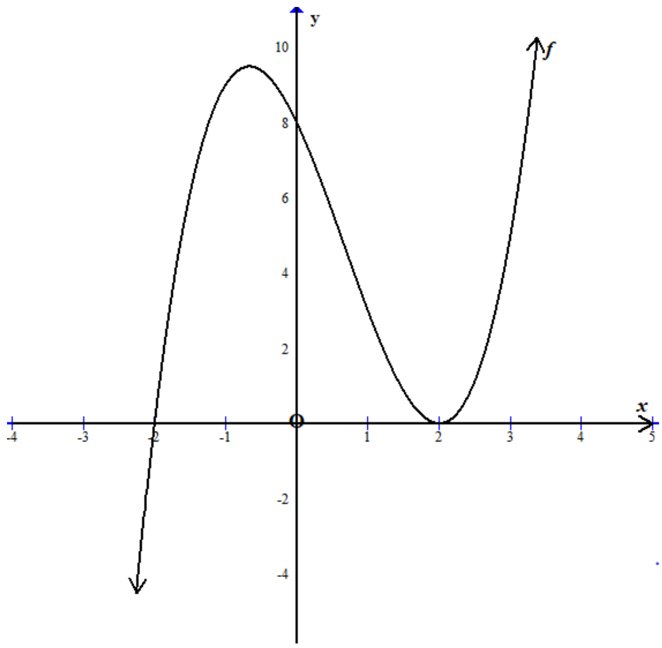
✓subst. into the correct formula

✓ *x*-values

✓both turning points

(5)

2.4



✓shape

✓both turning points

✓x-and y-intercepts.

(3)

3.1

$$y = a(x - x_1)(x - x_2)^2$$
$$2 = a(2 - 0)(2 - 3)^2$$
$$2 = a(2)(1)^2$$
$$a = 1$$
$$y = 1(x)(x - 3)^2$$
$$f(x) = x^3 - 6x^2 + 9x$$

✓subst.

✓ value of *a*

✓correct equation

(3)

3.2

$$y = x^3 - 6x^2 + 9x$$
$$f'(x) = 3x^2 - 12x + 9$$
$$x^2 - 4x + 3 = 0$$
$$(x - 3)(x - 1) = 0$$
$$x = 3 / x = 1$$
$$y = (3)^3 - 6(3)^2 + 9(3) = 0$$
$$y = (1)^3 - 6(1)^2 + 9(1) = 4$$
$$\therefore \mathrm{A}(1;4)$$

✓ derivative
✓ equating to zero

✓ factors
✓ *x*-values

✓ correct turning point

(5)

3.3 $x \geq 0$

✓answer (1)

4.1 $y=-(x+3)(x+1)(x-2)$

$y=-(x^3+2x^2-5x-6)$

$=-x^3-2x^2+5x+6$

$a=-2;\ b=5;\ c=6$

✓subst.

✓simplification

✓answers. (3)

4.2 $f^{/}(x)=-3x^2-4x+5$

$3x^2+4x-5=0$

$$x=\frac{-4\pm\sqrt{4^2-4(3)(-5)}}{2(3)}$$

$x=0,79 \quad or \quad x=-2,11$

$y=-(0,79)^3-2(0,79)^2+5(0,79)+6=8,20$

$y=-(-2,11)^3-2(-2,11)^2+5(-2,11)+6=-4,06$

$\therefore T(0,79;8,20)$ and $T(-2,11;-4,06)$

✓derivative

✓equating to zero

✓subst. into correct formula

✓ *x*-values

✓both turning points(5)

4.3 $x\le -3$ and $-1\le x\le 2$

✓ $x\le -3$

✓ $-1\le x\le 2$ (2)

4.4 $f^{/}(x)=-3x^2-4x+5$

$f'(2)=-3(2)^2-4(2)+5$

$=-12-8+5$

$\therefore m=-15$

✓derivative and subst.

✓ gradient (2)

5	$\frac{dy}{dx}=3x^2+8x+2$	✓derivative
	$m=3(-2)^2+8(-2)+2$	
	$=-2$	✓gradient
	$y=(-2)^3+4(-2)+2(-2)-3$	✓subst.
	$y=1$	✓y-value
	$(-2;1)$ and $m=-2$	✓subst. into correct formula
	equation of tangent:	
	$y-y_1=m(x-x_1)$	
	$y-1=-2(x-(-2))$	
	$y=-2x-3$	✓equation (6)
6	$f^{/}(x)=3x^2-3$	✓derivative
	$3x^2-3=24$	✓equate to 24
	$3x^2=27$	
	$x^2=9$	
	$x=\pm 3$	
	$\therefore r=3, \quad r>0$	✓value of r
	$y=(3)^3-3(3)+2$	
	$y=20$	
	$\therefore t=20$	✓value of t
		(4)

Optimisation

1.1	$s(0)=2(0)-25(0)+300$ $=300\text{m}$	✓subst. ✓answer	(2)
1.2	$s^{/}(t)=4t-25$ $s^{/}(5)=4(5)-25$ $=-5m/s$	✓subst. ✓answer.	(2)
1.3	$4t=25$ $t=\frac{25}{4}s$ $s\left(\frac{25}{4}\right)=2\left(\frac{25}{4}\right)^2-25\left(\frac{25}{4}\right)+300$ $=156{,}25\text{m}$	✓value of t ✓subst. ✓answer.	(3)
2.1	$\text{T}^{/}(t)=3-1{,}6t$	✓answer	(1)
2.2	$3-1{,}6t=0$ $-1{,}6t=-3$ $t=1{,}88\text{s}$	✓equating to zero ✓value of t.	(2)
3.1	$\frac{d\text{V}}{dr}=4\pi r^2$ $\frac{d\text{V}}{d(15)}=4\pi(15)^2$ $=900\pi$	✓derivative ✓subst. ✓answer.	(3)
3.2	$\frac{d\text{V}}{dr}=8\pi r$ $\frac{d\text{V}}{d(15)}=8\pi(15)$ $=120\pi$	✓derivative ✓subst. ✓answer.	(3)
4.1	$\text{V}=lbh$ $=x(2x)(60-2x)$ $=-120x^2-4x^3\ cm^3$	✓correct formula ✓subst. ✓answer.	(3)

4.2 $V'(x) = 0$ at maximum volume

$V'(x) = -120x^2 - 4x^3 \ cm^3$ ✓derivative

$0 = 240x - 120x^2$ ✓equating to zero

$12x(20 - x) = 0$ ✓common factor

$x \neq 0$ or $x = 20$ ✓value of x

$Dimensions = 20 \times 2(20) \times (60 - 2(20))$

$= 20 \times 40 \times 20$ ✓dimension (5)

4.3 $V = -120x^2 - 4x^3$

$= -120(20)^2 - 4(20)^3$ ✓subst.

$= 16000 \ cm^3$ ✓volume (2)

5.1 $\tan 60^0 = \dfrac{h}{r}$ ✓correct ratio

$\sqrt{3}.r = h$ ✓ $\sqrt{3}$

$r = \dfrac{h}{\sqrt{3}}$ ✓ value of r. (3)

5.2 $V = \dfrac{1}{3}\pi r^2 h$

$= \dfrac{1}{3}\pi\left(\dfrac{h}{\sqrt{3}}\right)^2 .h$ ✓subst.

$= \dfrac{1}{3}\pi.\dfrac{h^3}{3}$ ✓ $\dfrac{\pi h^3}{9}$

$= \dfrac{\pi h^3}{9}$ ✓subst.

$\dfrac{dV}{dh} = \dfrac{1}{3}\pi h^2$

$\dfrac{dV}{d(9)} = \dfrac{1}{3}\pi(9)^2$

$= 27\pi$ ✓answer (4)

6.1. $2x + y = 80$ ✓ 80

$y = 80 - 2x$ ✓ $80 - 2x$ (2)

6.2 $A = x.y$		
$= x(80-2x)$		
$= 80x - 2x^2$	✓ area	
$\dfrac{dA}{dx} = 80 - 4x$	✓derivative	
$80 - 4x = 0$		
$4x = 80$		
$x = 20\,\text{m}$	✓*x*-value	
$y = 80 - 2(20)$		
$y = 40\,\text{m}$	✓*y*-value.	(4)

Solutions to Integration Practice questions

1	$\int (x+3)^2\,dx$ $= \int \left(x^2 + 6x + 9\right) dx$ $= \dfrac{x^3}{3} + 3x^2 + 9x + \text{C}$	✓ simplification ✓ $\dfrac{x^3}{3}$ ✓ $3x^2$ ✓ $9x$ ✓ C (5)
2	$\int \sqrt{x}\left(x + x^2\right) dx$ $= \int x^{\frac{1}{2}}\left(x + x^2\right) dx$ $= \int \left(x^{\frac{3}{2}} + x^{\frac{5}{2}} \right) dx$ $= \dfrac{x^{\frac{5}{2}}}{\frac{5}{2}} + \dfrac{x^{\frac{7}{2}}}{\frac{7}{2}} + \text{C}$ $= \dfrac{2x^{\frac{5}{2}}}{5} + \dfrac{2x^{\frac{7}{2}}}{7} + \text{C}$	✓ $x^{\frac{1}{2}}$ ✓ simplification ✓ $\dfrac{x^{\frac{5}{2}}}{\frac{5}{2}}$ ✓ $\dfrac{x^{\frac{7}{2}}}{\frac{7}{2}} + \text{C}$ (4)
3	$\int \left(\dfrac{x^2 - 9}{x - 3} \right) dx$ $= \int \left(\dfrac{(x-3)(x+3)}{x-3} \right) dx$ $= \int (x+3)\,dx$ $= \dfrac{x^2}{2} + 3x + \text{C}$	✓ factors ✓ simplification ✓ $\dfrac{x^2}{2}$ ✓ $3x + \text{C}$ (4)

4	$\int\left(\sqrt[3]{x^2}+\frac{3}{x^2}\right)dx$ $=\int\left(x^{\frac{2}{3}}+3x^{-2}\right)dx$ $=\frac{x^{\frac{5}{3}}}{\frac{5}{3}}+\frac{3x^{-1}}{-1}+C$ $=\frac{3x^{\frac{5}{3}}}{5}-\frac{3}{x}$	✓ simplifying surd ✓ simplifying fraction ✓ $\frac{x^{\frac{5}{3}}}{\frac{5}{3}}$ ✓ $\frac{3x^{-1}}{-1}+C$ (4)
5	$\int\left(\frac{2x}{3}-\frac{5}{x^4}\right)dx$ $=\int\left(\frac{2}{3}x-5x^{-4}\right)dx$ $=\frac{2}{3}\cdot\frac{x^2}{2}-\frac{5x^{-3}}{-3}+C$ $=\frac{x^2}{3}+\frac{5}{3x^3}+C$	✓ $5x^{-4}$ ✓ $\frac{x^2}{3}$ ✓ ✓ $\frac{5}{3x^3}+C$ (4)
6	$\int\left(-\frac{6}{x}+\frac{1}{2}\pi p\right)dx$ $=-6\ln x+\frac{1}{2}\pi px+C$	✓ $-6\ln x$ ✓ $\frac{1}{2}\pi px+C$ (2)
7	$\int\left(5^x-m^4\right)dx$ $=\frac{5^x}{\ln x}-m^4x+C$	✓ $\frac{5^x}{\ln x}$ ✓ m^4x+C (2)
8	$\int\left(x^{-2}+\frac{1}{x}\right)dx$ $=-x^{-1}+\ln x+C$	✓ $-x^{-1}$ ✓ $\ln x$ ✓ C (3)

9	$\int\left(x^{\frac{1}{3}} - 5x^4\right)dx$ $= \dfrac{x^{\frac{4}{3}}}{\frac{4}{3}} - \dfrac{5x^5}{5} + C$ $= \dfrac{3x^{\frac{4}{3}}}{4} - x^5 + C$	✓ $\dfrac{3x^{\frac{4}{3}}}{4}$ ✓ $- x^5 + C$ (2)

Solutions to Practice Questions

1	$A = -\int \left(2x^3 - 4\right) dx$ $= -\left(\dfrac{x^4}{2} - 4x\right)\Bigg]_{-1}^{2}$ $= -\left[\left(\dfrac{(2)^4}{2} - 4(2)\right) - \left(\dfrac{(-1)^4}{2} - 4(-1)\right)\right]$ $\therefore$ A = 4,5 square units/*kwadraat eenhede* **OR/*OF*** $A = \int \left(2x^3 - 4\right) dx$ $= \left(\dfrac{x^4}{2} - 4x\right)\Bigg]_{-1}^{2}$ $= \left[\left(\dfrac{(2)^4}{2} - 4(2)\right) - \left(\dfrac{(-1)^4}{2} - 4(-1)\right)\right]$ $= -4,5$ $\therefore$ A = 4,5 square units /*kwadraat eenhede*	✓Area notation using intergrals/ *Area-notasie met gebruik van integrale* **M** ✓ $\dfrac{x^4}{2}$ **A** ✓ $-4x$ **A** ✓✓ **SF** **CA** ✓bounded area/ *begrensde oppervaklte* **CA** **OR/*OF*** ✓Area notation using intergrals/ *Area-notasie met gebruik van integrale* **M** ✓ $\dfrac{x^4}{2}$ **A** ✓ $-4x$ **A** ✓✓ **SF** **CA** ✓ bounded area/ *begrensde oppervaklte* **CA** (6)

3. EXAMINATION TIPS

- Always have relevant tools (Calculator, Mathematical Set, etc.)
- Read the instructions carefully.
- Thoroughly go through the question paper, check questions that you see you are going to collect a lot of marks, start with those questions in that order because you are allowed to start with any question but finish that question.
- Write neatly and legibly.

4. REFERENCES

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