GRADE 12

Technical Mathematics

SELF STUDY GUIDE

BOOK 3

1. TRIGONOMETRY
2. EUCLIDEAN GEOMETRY

basic education
Department: Basic Education
REPUBLIC OF SOUTH AFRICA
Second Chance Matric Support Programme
2030 NDP
NATIONAL EDUCATION COLLABORATION TRUST
ETDP SETA
WOZA MATRICS 2021 CATCH UP
STAY SAFE PROTECT SOUTH AFRICA
TOGETHER WE CAN BEAT THE CORONAVIRUS

(i) INTRODUCTION

The declaration of COVID-19 as a global pandemic by the World Health Organisation led to the disruption of effective teaching and learning in many schools in South Africa. The majority of learners in various grades spent less time in class due to the phased-in approach and rotational/ alternate attendance system that was implemented by various provinces. Consequently, the majority of schools were not able to complete all the relevant content designed for specific grades in accordance with the Curriculum and Assessment Policy Statements in most subjects.

As part of mitigating against the impact of COVID-19 on the current Grade 12, the Department of Basic Education (DBE) worked in collaboration with subject specialists from various Provincial Education Departments (PEDs) developed this Self-Study Guide. The Study Guide covers those topics, skills and concepts that are located in Grade 12, that are critical to lay the foundation for Grade 12. The main aim is to close the pre-existing content gaps in order to strengthen the mastery of subject knowledge in Grade 12. More importantly, the Study Guide will engender the attitudes in the learners to learning independently while mastering the core cross-cutting concepts.

(ii) HOW TO USE THIS SELF STUDY GUIDE?

- This study guide covers two topics, namely Differential Calculus and Integration.
- In the 2021, there are three Technical Mathematics Booklets. This one is Booklet 2. Booklet 1 covers Algebra as well as Functions and Graphs while Booklet 3 covers Trigonometry and Euclidean Geometry.
- For each topic, sub-topics are listed followed by the weighting of the topic in the paper where it belongs. This booklet covers the two topics mentioned which belong to Technical Mathematics Paper 1
- Definitions of concepts are provided for your understanding
- Concepts are explained first so that you understand what action is expected when approaching problems in that particular concept.
- Worked examples are done for you to follow the steps that you must follow to solve the problem.
- Exercises are also provided so that you have enough practice.
- Selected Exercises have their solutions provided for easy referral/ checking your correctness.
- More Exam type questions are provided.

1. Trigonometry

In this topic learners must be able to:

- Apply trig ratios in solving right angled triangles in all four quadrants and by making use of diagrams.
- Apply the sine, cosine, and area rule.
- Apply reciprocals of the three basic trigonometric ratios.
- Use the calculators where applicable.
- Solve problems in two dimensions using sine, cosine, and area rule.
- Draw graphs defined by $y = a\sin x, y = a\cos x, y = \sin kx, y = \cos kx$ and $y = a\tan x$.
- Draw the graphs of the functions of $y = a\sin(x + p)$ and $y = a\cos(x + p)$
- Solve Trigonometric equations
- Apply identities
- Rotating vectors in developing sine and cosine curves.

USING TRIGONOMETRIC RATIOS TO SOLVE RIGHT ANGLED TRIANGLES IN ALL FOUR QUADRANTS.

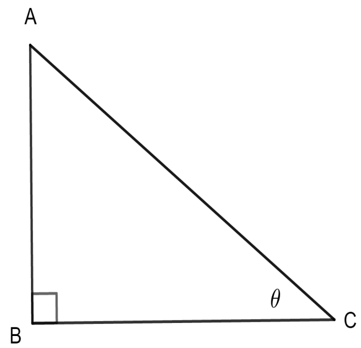
- To calculate the sizes of unknown sides and angles while provided with the sizes of the other dimensions.

PRIOR KNOWLEDGE

- Fractions (basic operations)
- Interval notation / set builder notation
- Calculator usage

PYTHAGORAS THEOREM

- In a right angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides
- Given ΔABC :



$$AC^2 = BC^2 + AB^2$$

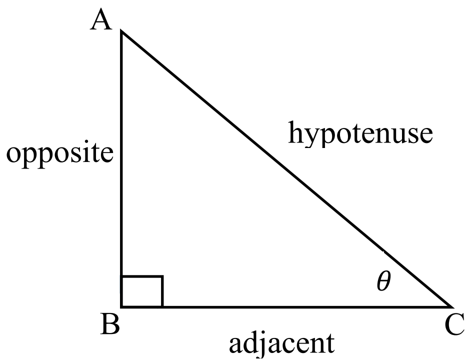
- Hypotenuse side is the side opposite a 90^0 angle. The side is the longest side of a right angled triangle.
- AC is the hypotenuse side
- The theorem is used to calculate the length of an unknown side of right angled a triangle, given the other two sides.

TRIGONOMETRIC RATIOS

Trigonometric Ratios and Trigonometric Reciprocals Ratios

Trigonometric Ratios	Trigonometric Reciprocals Ratios
$\sin\theta$	$\cos ec\ \theta = \frac{1}{\sin\theta}$
$\cos\ \theta$	$\sec\theta = \frac{1}{\cos\ \theta}$
$\tan\theta$	$\cot\theta = \frac{1}{\tan\theta}$

In a right-angled triangle, we can define trigonometric ratios as follows:



Pythagoras Theorem:

$AC^2 = AB^2 + BC^2$

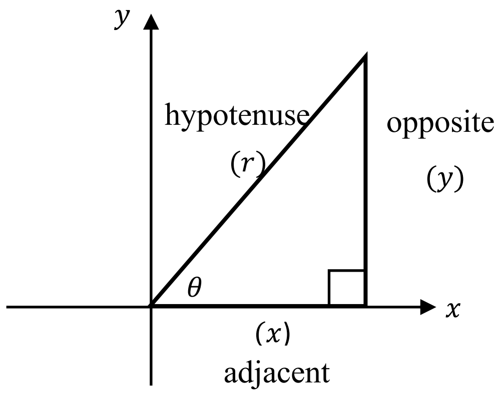
Trigonometric ratios

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

A right-angled triangle can be drawn in a Cartesian plane.



Pythagoras Theorem:

$$x^2 + y^2 = r^2$$

$$\sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x}$$

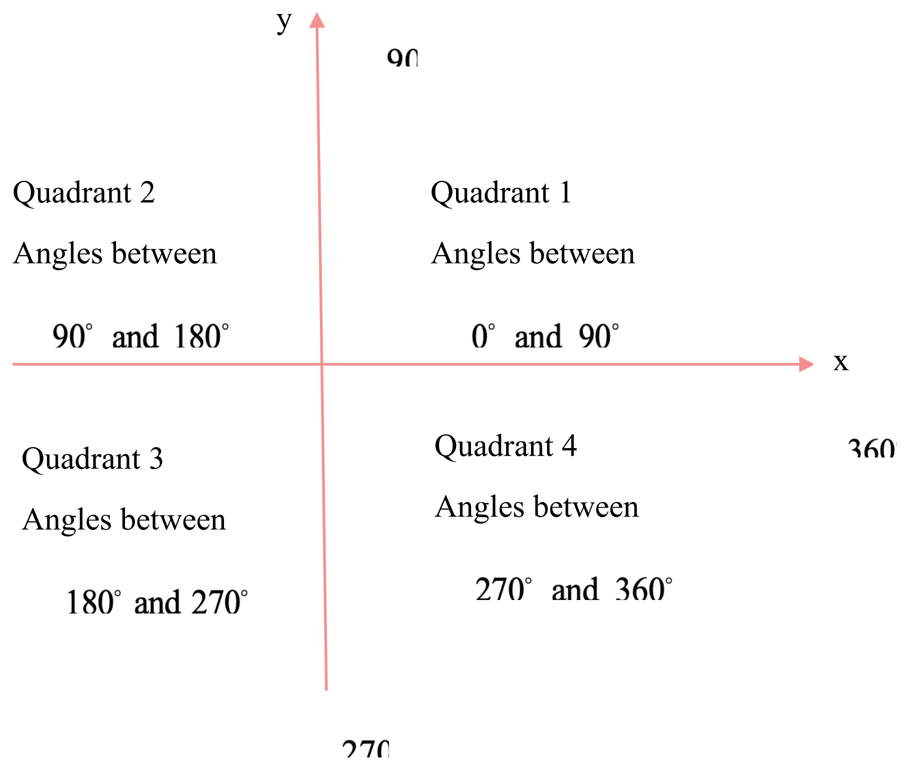
RECIPROCAL RATIOS

$$\operatorname{cosec}\theta = \frac{r}{y}$$

$$\sec\theta = \frac{r}{x}$$

$$\cot\theta = \frac{x}{y}$$

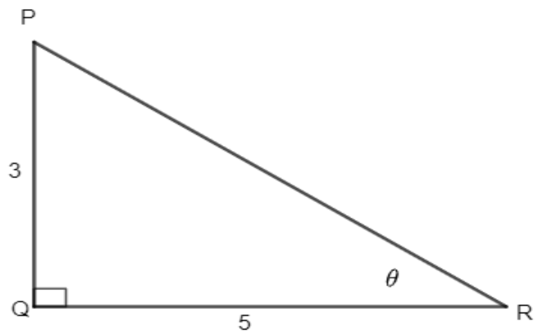
RESTRICTIONS OR INTERVALS



Complete the table below:

Restrictions or Intervals	Quadrant/s	How to read restrictions
1 $0° < \theta < 90°$	1^{st}	θ is greater than $0°$ and less than $90°$
2 $90° < \beta < 270°$	2^{nd} and 3^{rd}	β is greater than 90^0 and less than 270°
3 $90° < \alpha < 180°$		
4 $180° < \alpha < 360°$		
5 $90° < \alpha < 360°$		
6 $270° < \alpha < 360°$		

WORKED EXAMPLES

1	Give triangle PQR, PQ =3, QR=5  Determine the following, and leave your answers as in surd form	
	1.1	PR $PR^2 = PQ^2 + QR^2$ Theorem of Pythagoras $= 3^2 + 5^2$ Substitution $= 29$ Simplification $PR = \sqrt{29}$ answer
	1.2	$\sin\theta$ $\sin\theta = \frac{PQ}{PR} = \frac{3}{\sqrt{29}}$ answer
	1.3	$\tan\hat{P}$ $\tan\hat{P} = \frac{QR}{PQ} = \frac{5}{3}$ answer
	1.4	$\sec\theta + \tan\hat{P}$ $= \frac{\sqrt{29}}{3} + \frac{5}{3}$ $= \frac{\sqrt{29} + 5}{3}$

2 Given where $180° < \beta < 360°$. Calculate without using a calculator, the value of

2.1 $\sin\beta$ (leave your answer in surd form)

Step 1 : Identify the quadrant where $\cos\beta$ is negative

2nd and 3rd quadrant are possible

Step 2: Use the given restriction to decide on the correct quadrant

The restriction eliminates the 2nd quadrant since

$180° < \beta < 360°$ represents the 3rd and 4th quadrant

Conclusion: 3rd quadrant is where both statements are true

Step 3: Draw a sketch to indicate the quadrant

Step 4: Calculate the unknown variable using Pythagoras Theorem

$x^2 + y^2 = r^2$ Pythagoras Theorem

$(-2)^2 + (y)^2 = (3)^2$ substitution

$4 + y^2 = 9$

$y^2 = 9 - 4$

$y^2 = 5$

$y = -\sqrt{5}$ answer (correct quadrant)

Step 5: Answer questions based on calculations

$\sin\beta = \dfrac{\text{opposite}}{\text{hypotenuse}} = -\dfrac{\sqrt{5}}{3}$ substitution

	2.2	$5\cot^2\beta$ $=\dfrac{5}{\tan^2\beta}$ $=5\times\dfrac{1}{\left(\dfrac{-\sqrt{5}}{-2}\right)}$ $=4$	

PRACTICE QUESTIONS

1 Given $\tan\alpha=\dfrac{3}{4}$ where $\alpha[0°;90°]$. With the use of a sketch and without a calculator calculate:

1.1 $\sin\alpha$ (2)

1.2 $1-2\sin\alpha\cdot\cos\alpha$ (3)

1.3 $\cos^2(90°-\alpha)-1$ (3)

2 If $6\cos A+3=0$ and $A[180°;360°]$,without a calculator determine the numerical value of $3\tan^2 A+\sin A$ (5)

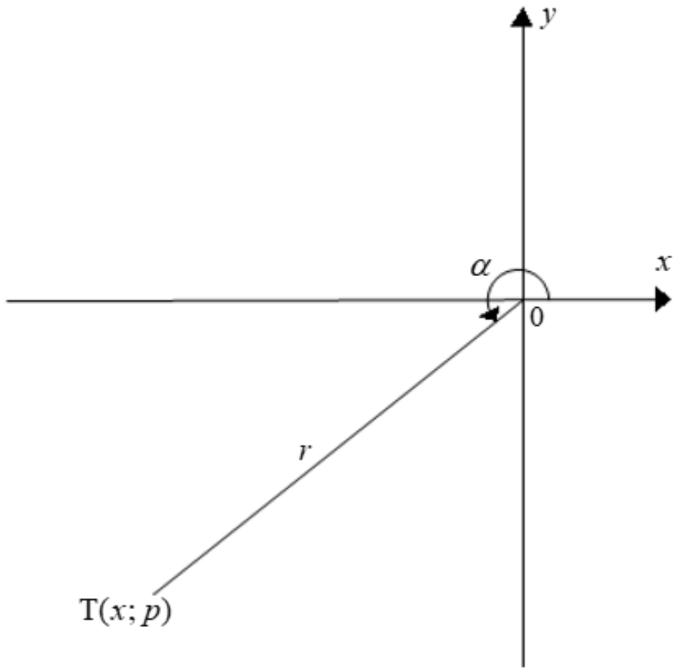
3 If $\sin 37°=p$, with the aid of a diagram and without using a calculator determine the following in terms of p

3.1 $\cos 53°$ (3)

3.2 $\tan 37°$ (3)

3.2 $\cos 143°$ (3)

4 In the diagram below, $\mathrm{T}\ (x; p)$ is a point in the third quadrant and it is given that $\sin\alpha = \dfrac{p}{\sqrt{1 - p^2}}$



4.1	Show that $x = -1$	(2)
4.2	Write $\cos(180° + \alpha)$ in terms of *p*.	(2)

Trigonometric expressions and equations

Trigonometric expressions: using a calculator to evaluate expressions

1	If $x = 159,3°$ and $x = 36,7°$; determine the following without the use of a calculator:	
	1.1	$\sin(x - y)$ $= \sin(159,3° - 36,7°)$ substitution $= 0,84$ answer
	1.2	$\sin x - \sin y$ $= \sin 159,3° - \sin 36,7°$ $= -0,244$
	1.3	cosec x $= \text{cosec } 159,3^0$ $= 2,83$
	1.4	$\cot 2y$ $= \cot 2(36,7°)$ $= 0,78$
2	If $\alpha = 1,4\pi$ and $\beta = 2,3\pi$, determine:	
	2.1	$\sec(\alpha + \beta)$ $= \sec 3,7\pi$ $= 1,70$
	2.2	$\cos^2\alpha + \sin^2\alpha$ $= \cos^2 1,4\pi + \sin^2 1,4\pi$ $= 1$

Trigonometric equations

Prior knowledge

- Solving simple equations
- Calculator usage

Worked examples

1	Solve for an unknown in the equations that follow	
	1.1	$\cos x = 0{,}34 \quad x \in [0°;180°]$ $x = \cos^{-1} 0{,}34$ answer
	1.2	$2\cos 2\theta = -1;\ 2\theta \in [0°;360°]$ $\cos 2\theta = -\frac{1}{2}$ isolating a trig ratio $\cos^{-1}\left(\frac{1}{2}\right) = 60°$ reference angle $2\theta = 180° - 60°$ 2nd quadrant $2\theta = 120°$ $\theta = 60°$ answer or $2\theta = 180° + 60°$ Third quadrant $2\theta = 240°$ $\theta = 120°$ **Note: cosine is negative in the 2nd and 3rd quadrants**
2		$2\cos\theta - 3\sin\theta = 0$ $\frac{2\cos\theta}{\cos\theta} - \frac{3\sin\theta}{\cos\theta} = 0$ use of $\cos\theta$ $2 - 3\tan\theta = 0$ $\tan\theta = \frac{2}{3}$ single trig ratio Reference angle : $\theta = 33{,}7°$ ref angle Using the CAST RULE θ also falls in 1st and 3rd quadrant $\theta = 33{,}7°$ or $\theta = 180° - 33{,}7° = 146{,}3°$ both answers

PRACTICE QUESTIONS

	Solve the following unknown	
1	$\cos x = -0{,}349\ ;\ 0° \leq x \leq 360°$	(3)
2	$\tan\theta = 5\sin 71°$	(3)
3	Solve for θ correct to TWO decimal places, if $\frac{4}{3}\sin\theta = \tan 43°$	(3)
4	$\tan 2x = 2{,}114\ ;\ 2x \in [0°;180°]$	(3)
5	Determine the value of $\theta \in [90°;360°]$ if $7\sin\theta - 5 = 0$	(4)

Trigonometric identities

The following identities can be applied in simplifying trig expressions

$\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta = 1 - \cos^2\theta$

$\cos^2 = 1 - \sin^2\theta$

$1 + \tan^2\theta = \sec^2\theta$

$\tan^2\theta = \sec^2\theta - 1$

$1 + \cot^2\theta = \cos ec^2\theta$

$\cot^2\theta = \cos ec^2\theta - 1$

S **Reduction formulae for $(180° \pm \theta$ and $360° \pm \theta)$** **A**

$\sin(180° - \theta) = \sin\theta$ $\cos(180° - \theta) = -\cos\theta$ $\tan(180° - \theta) = -\tan\theta$ $180° -$	θ $\sin(360° + \theta) = \sin\theta$ $\cos(360° + \theta) = \cos\theta$ $\tan(360° + \theta) = \tan\theta$ $360° + \theta$
$180° +$ $\sin(180° + \theta) = -\sin\theta$ $\cos(180° + \theta) = -\cos\theta$ $\tan(180° + \theta) = \tan\theta$	$360° -$ $\sin(360° - \theta) = -\sin\theta$ $\cos(360° - \theta) = \cos\theta$ $\tan(360° - \theta) = -\tan\theta$

T **C**

Reduction formula are also used to simplify expressions and prove identities

Worked examples

1 Simplify the following

1.1

$$\frac{\sin(360° - \theta)\cdot\cos(\pi + \theta)}{\sin(180° - \theta)\cdot\sin\theta}$$

$$= \frac{-\sin\theta\cdot-\cos\theta}{\sin\theta\cdot\sin\theta}$$ single trig ratios

$$= \frac{\cos\theta}{\sin\theta}$$ simplification

$$= \tan\theta$$ answer

1.2

$$\frac{\sin^2 x - 1}{\cos^2 x}$$

$$= \frac{-(1 - \sin^2 x}{\cos^2 x}$$

$$= \frac{-\cos^2 x}{\cos^2 x}$$ identity in numerator

$$= -1$$ answer

2 Prove the following identities

2.1

$$\frac{\cot\theta\cdot\sec\theta}{\cos ec\theta} = 1$$

$$\text{L.H.S.} = \frac{\cot\theta\cdot\sec\theta}{\cos ec\theta}$$

$$= \frac{\frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\cos\theta}}{\frac{1}{\sin\theta}}$$ $\tan\theta$ identity

$$= \frac{\frac{1}{\sin\theta}}{\frac{1}{\sin\theta}}$$ simplification

$$= 1$$

	2.2	$\frac{\sin\theta - \sin\theta.\cos\theta}{\cos\theta - (1-\sin^2\theta)} = \tan\theta$ $LHS = \frac{\sin\theta - \sin\theta.\cos\theta}{\cos\theta - (1-\sin^2\theta)}$ $= \frac{\sin\theta(1-\cos\theta)}{\cos\theta - \cos^2\theta}$ $= \frac{\sin\theta(1-\cos\theta)}{\cos\theta(1-\cos\theta)}$ $= \frac{\sin\theta}{\cos\theta}$ $= \tan\theta$	common factor , $\cos^2\theta$ common factor
	2.3	$\frac{2\sin x\cdot\cos x(1+\tan^2 x)}{\tan x}$ $LHS = \frac{2\sin x.\cos x.\sec^2 x}{\tan x}$ $= \frac{2\sin x.\cos x.\frac{1}{\cos^2 x}}{\frac{\sin x}{\cos x}}$ $= \frac{\frac{2\sin x}{\cos x}}{\frac{\sin x}{\cos x}}$ $= 2$	square identity $\frac{1}{\cos^2 x}$; $\frac{\sin x}{\cos x}$ simplification

PRACTICE QUESTIONS

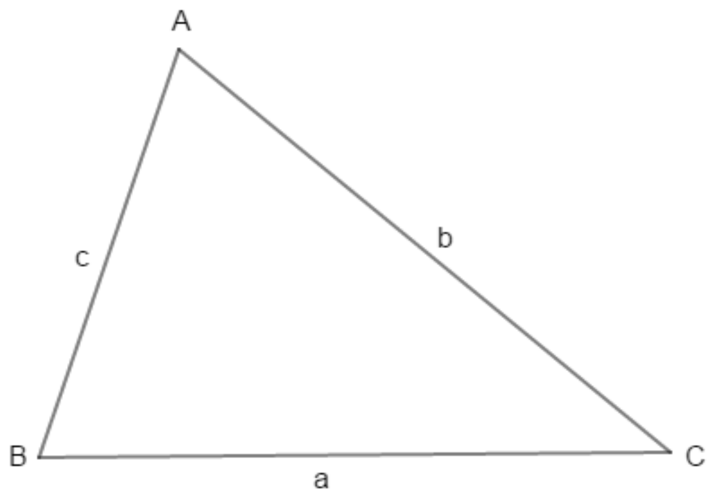
1	Simplify the following		
	1.1	$\cos x.\tan^2 x + \cos x$	(3)
	1.2	$\dfrac{\cos^2 x}{1-\sin x} - \sin x$	(4)
	1.3	$\dfrac{\tan x + \cot x}{\cos ecx}$	(5)
2	Prove the following identities		
	2.1	$3\cot^2 x\ (\tan^2 x + 1) = 3\text{cosec}^2\ x$	(4)
	2.2	$\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta} = \dfrac{1}{\sin\theta\cos\theta}$	(3)
	2.3	$\dfrac{\sin x}{1-\sin x} + \dfrac{\sin x}{1+\sin x} = \dfrac{2\tan x}{\cos x}$	(4)

Application of the sine, cosine and area rule

Triangles that are not right angled are solved using the above rules

The Sine rule

In any ΔABC : $\dfrac{a}{\sin \hat{A}} = \dfrac{b}{\sin \hat{B}} = \dfrac{c}{\sin \hat{B}}$ or $\dfrac{\sin \hat{A}}{a} = \dfrac{\sin \hat{B}}{b} = \dfrac{\sin \hat{C}}{c}$



The Sine Rule can be applied when:

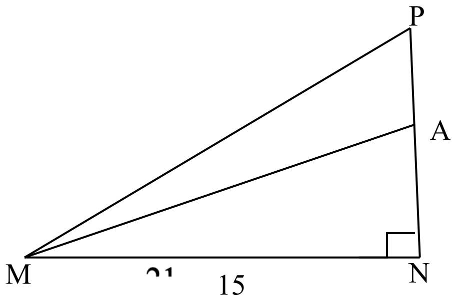
- Two sides and one angle are given. The given angle must be opposite one of the given sides.
- Two angles and one side of a triangle are given. The given side must be opposite one of the given angles.
- These are applied in the context of 2D and 3D problems

Worked examples

1

In the sketch below, Δ MNP is drawn having a right angle at N and MN = 15 units.

A is the midpoint of PN and $A\hat{M}N = 21°$



Calculate: (Correct your answer to 2 decimals)

1.1 The length of AN

Step 1: Analyse the given information

In triangle AMN:

Given an angle and its adjacent side

Unknown: opposite side

Step 2: Based on the given information, decide on the ratio to use

$$\tan 21° = \frac{AN}{15}$$

$$AN = 15 \times \tan 21°$$

$$AN = 5{,}76$$

1.2 If $PA = AN$, calculate the length of MP

$PA = 5{,}76$ and $PN = 11{,}52$

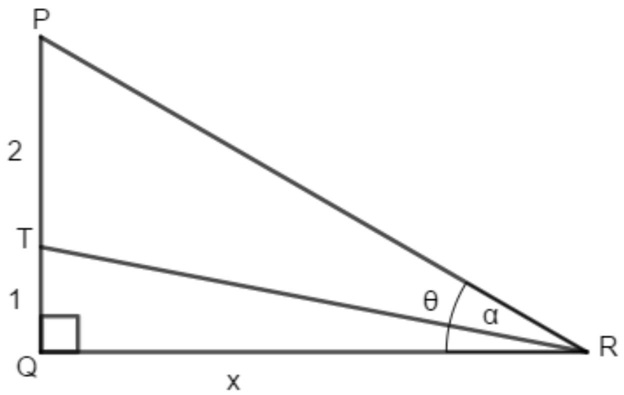
Unknown side: hypotenuse

$$PM^2 = PN^2 + MN^2$$

$$PM^2 = 11{,}52^2 + 15^2$$ Pythagoras

$$PM^2 = 357{,}71$$

$$PM = 18{,}91$$ answer

	1.3	The size of $\hat{P}$ $\frac{\sin \hat{P}}{MN} = \frac{\sin \hat{N}}{MP}$ $\frac{\sin \hat{P}}{15} = \frac{\sin 90^0}{18,91}$ sine Rule $\sin \hat{P} = \frac{15 \times \sin 90^0}{18,91}$ $\sin \hat{P} = 0,793$ simplification $\hat{P} = 52,48^0$ answer
2	In the diagram alongside .PT is a 2 metre high screen which is 1 metre above eye level of a learner standing at R. $QR = x metres$. The angle of elevation of P, top of the screen, from R is θ, i.e. $P\hat{R}Q = \theta$ and $P\hat{R}T = \alpha$  	
	2.1	Express $T\hat{R}Q = \theta$ in terms of θ and α $T\hat{R}Q = \theta = \alpha - \theta$
	2.2	Express TR in terms of x $\cos\ (\alpha - \theta) = \frac{x}{TR}$ $TR = \frac{x}{\cos\ (\alpha - \theta)}$
	2.3	Express PR in terms of x $PR^2 = 1^2 + x^2$ $PR = \sqrt{1 + x^2}$

Cosine rule

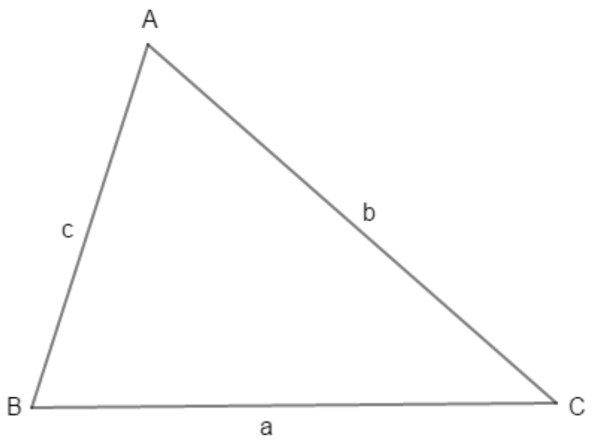
The cosine rule is used:

If the triangle is not right angled and:

- Three sides are given
- Two sides and an angle must be given provided the given angle is an included angle

The cosine rule for triangle ABC is given by:

- $a^2 = b^2 + c^2 - 2bc\cos \mathrm{A}$
- $b^2 = a^2 + c^2 - 2ac\cos \mathrm{B}$
- $c^2 = a^2 + b^2 - 2ab\cos \mathrm{C}$

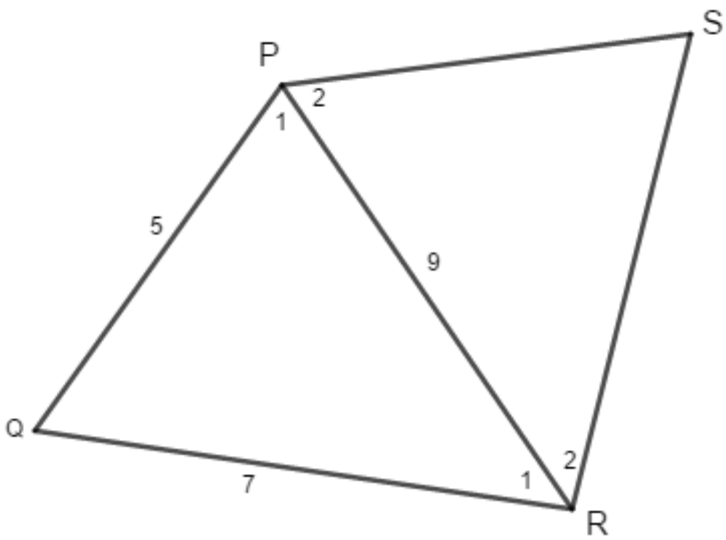


$$\cos \mathrm{A} = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos \mathrm{B} = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos \mathrm{C} = \frac{a^2 + b^2 - c^2}{2ab}$$

WORKED EXAMPLES

1		In the figure below P, Q, R, S are points on the circle. PQ = 5 units, QR = 7 units and PR = 9 units  Calculate the following, answer rounded off to one decimal digit	
	1.1	The size of $\hat{Q}$ Given the lengths of the 3 sides of a triangle: Use the cosine rule $PR^2 = PQ^2 + QR^2 - 2PQ \cdot QR\cos \hat{Q}$ Or $\cos \hat{Q} = \frac{p^2 + r^2 - q^2}{2pr}$ $6 = \frac{7^2 + 5^2 - 9^2}{2(7)(5)}$ $= -0,1$ $\hat{Q} = 95,74°$	(3)
	1.2	$\hat{S}$ if $\hat{S} + \hat{S} = 180°$ $\hat{S} = 84,26^0$	(2)

1.3 $\hat{R}_2$, if PS = 4

In triangle PSR: Two sides and one angle are known, use the sine rule

$$\frac{\sin \hat{R}_2}{4} = \frac{\sin \hat{S}}{9}$$

$$\sin \hat{R}_2 = \frac{4\sin 84{,}26^0}{9}$$

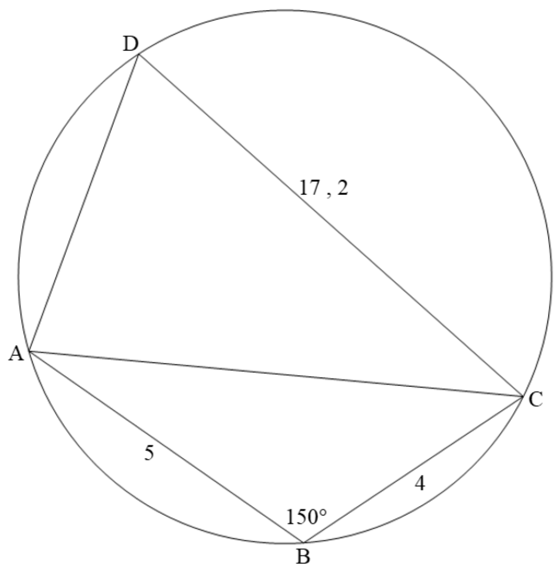
$$\sin \hat{R}_2 = 0{,}44$$

$$\hat{R}_2 = 26{,}25^0$$

(3)

2 In the accompanying cyclic quadrilateral AB = 5 units, BC = 4 units,

CD = 17 , 2 units and $\hat{B} = 150°$



Calculate:

2.1 Length of AC (4)

$$AC^2 = AB^2 + BC^2 - 2(AB.BC)\cos \hat{B}$$
$$= 5^2 + 4^2 - 2(5)(4)\cos 150^0$$
$$= 18{,}8$$

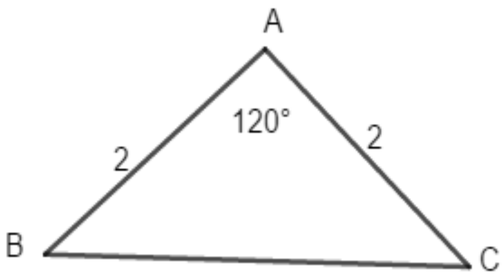
	2.2	Size of $\hat{D}$ $\hat{D} = 30°$	(2)
	2.3	Size of $C\hat{A}D$ $\frac{\sin C\hat{A}D}{17,2} = \frac{\sin 30°}{18,8}$ $\sin C\hat{A}D = \frac{17,2 \sin 30°}{18,8}$ $\sin C\hat{A}D = 0,457$ $C\hat{A}D = 27,22^0$	(3)

The area rule

Two sides and an included angle must be given in order to calculate the area of a triangle

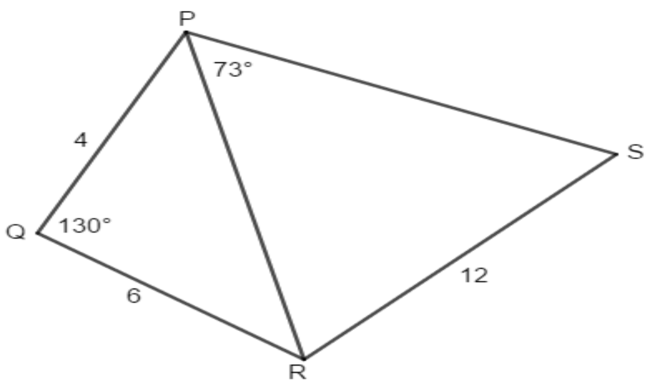
Worked examples

1 Without using a calculator, calculate the area of triangle ABC. Leave your answer in surd form.



$$\text{Area of } \Delta ABC = \frac{1}{2}(2)(2)\sin 120°$$
$$= 1$$

2 In the diagram, PQRS is a quadrilateral with $PQ = 4\text{cm}$, $RQ = 6\text{cm}$, $SR = 12\text{cm}$, $\hat{Q} = 130°$ and $R\hat{P}S = 73°$



2.1 Show that $PR = 9,1 \text{ cm}$ (2)

$$PR^2 = 4^2 + 6^2 - 2(4)(6)\cos 130°$$
$$= 16 + 36 - 48\cos 130°$$
$$= 82,85$$
$$\therefore PR = 9,1 \text{ cm}$$

2.2 Calculate the size of $\hat{S}$ rounded off to two decimal places. (2)

$$\frac{\sin\hat{S}}{9,1} = \frac{\sin 73^\circ}{12}$$

$$\sin\hat{S} = \frac{9,1\sin 73^\circ}{12}$$

$$= 0,725$$

$$\hat{S} = 48,5^\circ$$

2.3 Determine the area of ΔPQR rounded off to two decimal digits. (2)

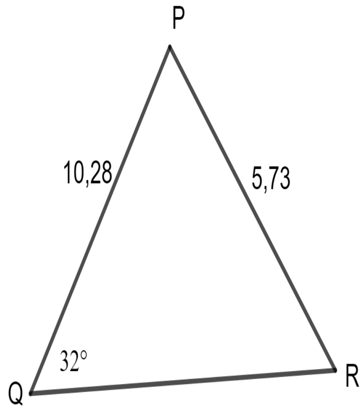
$$\text{Area of } \Delta ABC = \frac{1}{2}(4)(6)\sin 130^\circ$$

$$= 9,19$$

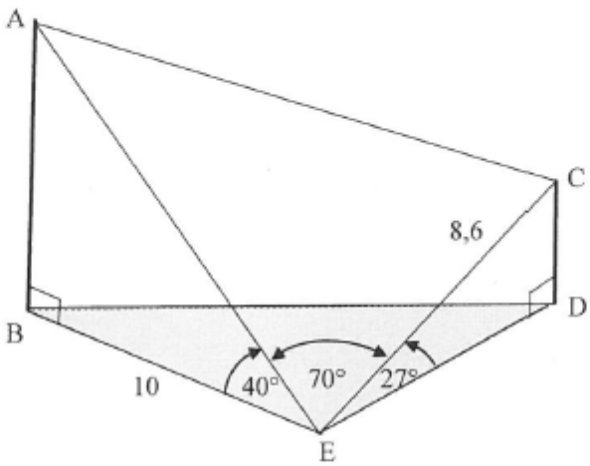
Practise questions

1 In the accompanying diagram $QP = 10,28$, $PR = 5,73$, $\hat{Q} = 32°$

Calculate $\hat{P}$. (3)



2 In the diagram, B, E and D, are points in the same horizontal plane. AB and CD are vertical poles. Steel cables AE and CE anchor the poles at E. Another steel cable connects A and C. $CE = 8,6\text{ m}$; $BE = 10\text{ m}$; $A\hat{E}B = 40°$ and $C\hat{E}D = 27°$



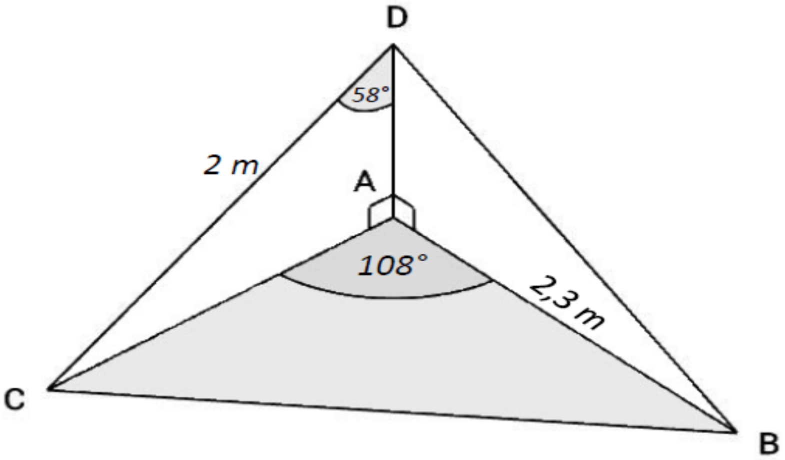
Calculate:

2.1 Height of pole CD. (2)

2.2 Length of cable AE. (2)

2.3 Length of cable AC (4)

3 The diagram below shows a vertical pole AD with points C and B on the same horizontal plane as A, the base of the pole. If $C\hat{A}D = 58°$, $C\hat{A}B = 30°$, CD = 2m and AB = 2,3 m



Calculate:

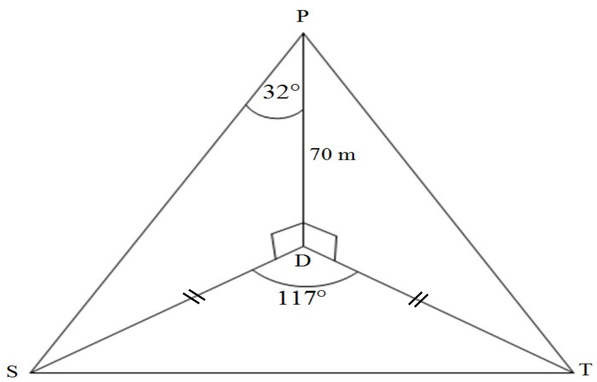
3.1 The length of AC (2)

3.2 The area of ΔABC (3)

3.3 The length of BC (3)

3.4 The size of $C\hat{D}B$ if BD = 2,5 m (4)

4 The diagram below shows the position of a helicopter at point P, which is directly above point D on the ground. Points S, D and T lie in the same horizontal plane, such that points S and T are equidistant from D. $PD = 70m$, $S\hat{T}D = 117^\circ$ and $S\hat{P}T = 32^\circ$



Calculate:

4.1 The distance SD (2)

4.2 The distance ST (3)

4.3 The area of ΔSDT (2)

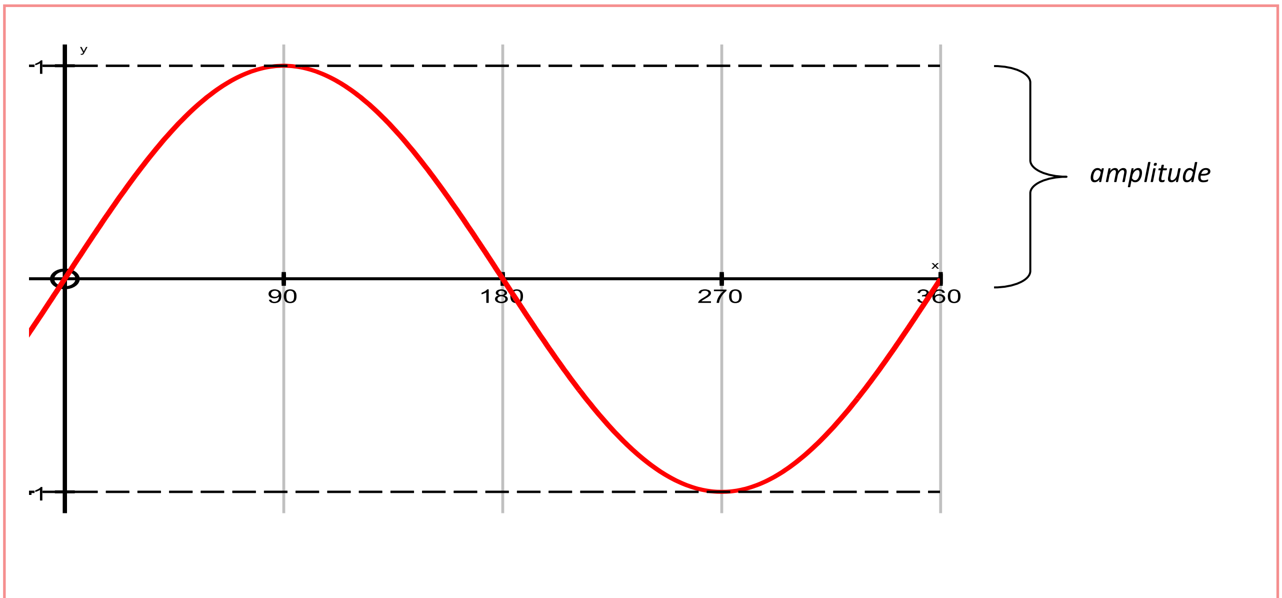
Trigonometric functions

Graph	Period	Amplitude	Domain	Range
$y=\sin x$	$360°$	1	$[0°,360°]$	$-1\le y\le 1$ or $y\in[-1,1]$
$y=\cos x$	$360°$	1	$[0°,360°]$	$-1\le y\le 1$ or $y\in[-1,1]$
$y=\tan x$	$180°$	undefined	$[0°,360°]$	Undefined

Sketching of the graph: $y=\sin x$

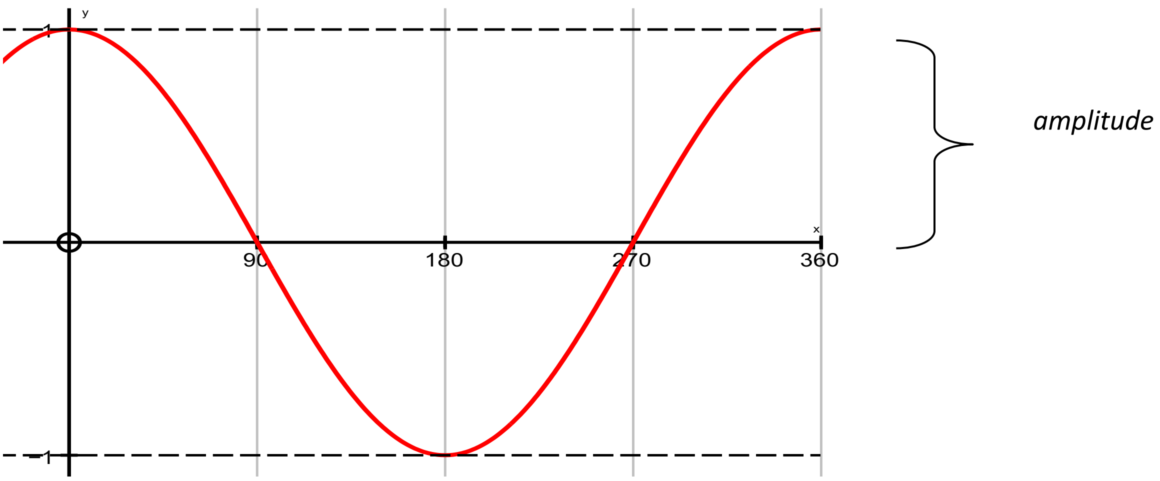
Using the table (Point by point plotting)

Y	$0°$	$90°$	$180°$	$270°$	$360°$
$y=\sin x$	0	1	0	-1	0



- $y=\sin x$ has a period of 360° because the curve repeats itself every 360°.
- This graph has a maximum of +1 and a minimum of –1.
- that the amplitude is 1.
- The range is $\{-1\le y\le 1\}$ or $\{-1;1\}$

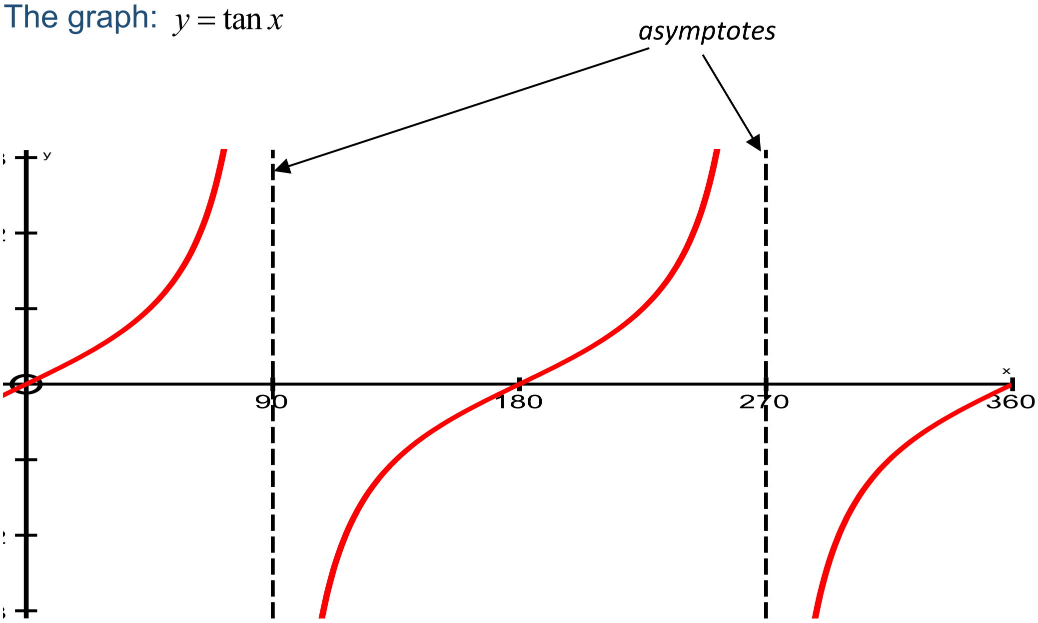
The graph: $y = \cos x$



- $y = \cos x$ has a period of 360° because the curve repeats itself every 360°.
- This graph has a maximum of +1 and a minimum of –1.
- The amplitude is 1.

The range is $\{-1 \leq y \leq 1\}$ or $\{-1;1\}$

The graph: $y = \tan x$



- $y = \tan x$ has a period of 180° because the curve repeats itself every 180°.
- This graph tends towards positive or negative infinity at the asymptotes and hence the range and amplitude are undefined.
- as the graph gets close to, say, 90° the graph gets steeper and steeper. It will never touch this line, or the lines $x = 90^\circ; x = 270^\circ$ etc.
- These asymptotes are 180° apart.
- always show the point $\left(45^\circ, 1\right)$ to give some idea of the scale on the vertical axis.
- indicate the asymptotes by means of dotted vertical lines.

The effect of parameters

$y = a\sin(kx \pm p) \pm q$	$y = a\cos(kx \pm p) \pm q$	$y = a\tan(kx \pm p) \pm q$

Key concepts:

For both functions of:

$y = a\sin(kx \pm p) \pm q$	$y = a\cos(kx \pm p) \pm q$
The effects of the following variables on the graph	
a	Affects the amplitude The sign of *a* affects the shape of the graph
k	Affects the $\text{period} = \frac{360^\circ}{k}$
p	Shifts the graph left and right
q	Shifts the graphs upward and downward

For tangent functions:

$y = a\tan(kx \pm p) \pm q$	
The effects of the following variables on the graph	
a	Affects the steepness of the graph The sign of *a* affects the shape of the graph
k	Affects the $\text{period} = \frac{180^\circ}{k}$
p	Shifts the graph left and right
q	Shifts the graphs upward and downward

WORKED EXAMPLES

1 Sketch the graph of $y = 3\sin 2x$ for $x \in [0°;360°]$

$y = 3\sin 2x$

Step 1: $\text{period} = \dfrac{360^\circ}{k}$

The value in front of *x* is *k* = 2

$\text{Period} = \dfrac{360°}{2}$

$\text{Period} = 180°$

Note: the graph will complete the cycle after $180°$ implying over $360°$ there will be two complete cycles(waves).

Step 2: Critical values (always divide the period by 4)

$\text{Critical value: } \dfrac{180^\circ}{4} = 45^\circ$

Note: These are the intervals to use in sketching the graph

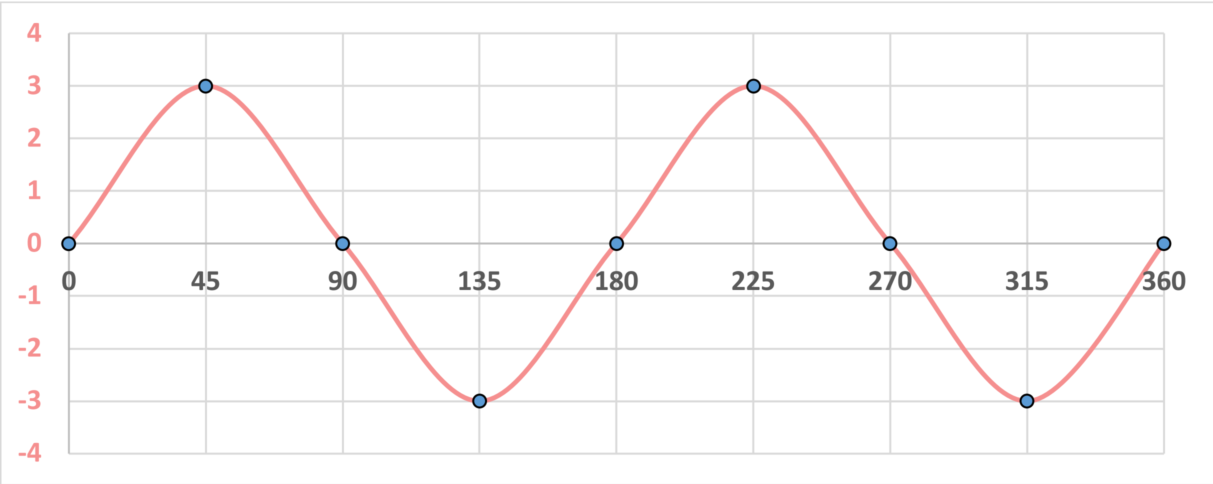
Step 3: Using a calculator to get the output values $y = 3\sin 2x$

Substitute the value of x in the bracket to get y values

x	45°	90°	135°	180°	225°	270°	315°	360°
y	3	0	-3	0	3	0	-3	0

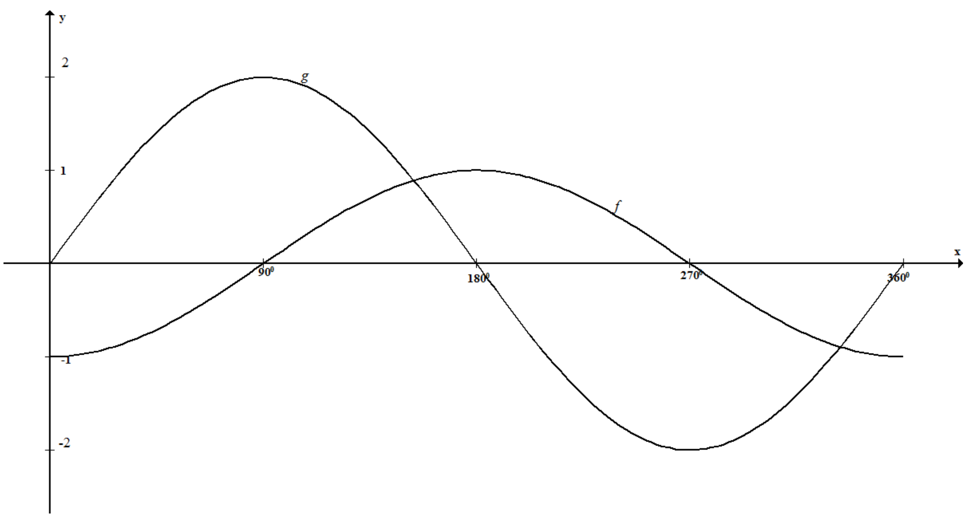
Step 4

Sketch the graph



2 Given the functions defined by $f(x) = -\cos x$ and $g(x) = 2\sin x, [0°;360°]$

2.1 Draw f and g on the same set of axes.



2.2 Write down the period of f.

360°

2.2 Write down the amplitude of g.

2

2.3 Write down the value(s) of x for which $f(x) \cdot g(x) \geq 0$

$x \in [90°; 180°]$

or

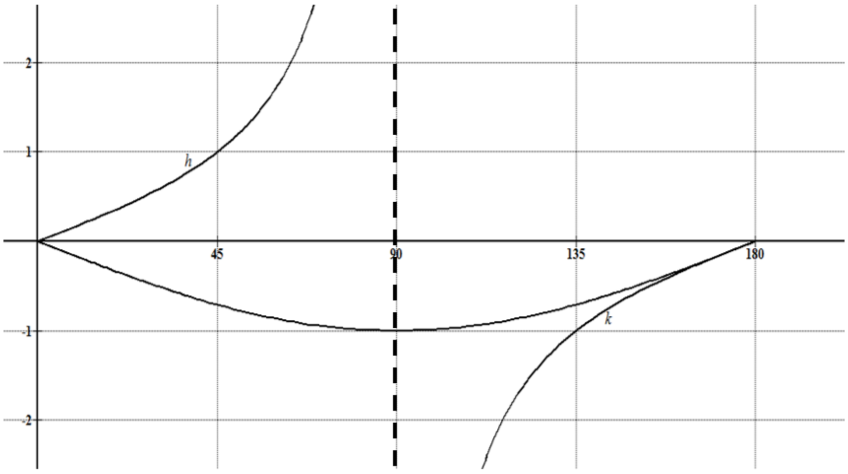
$x \in [270°; 360°]$

2.4 Write down the turning points of k if $k(x) = f(x + 60°)$

Note: The graph will experience a horizontal shift $60°$ to the left

$(120°; 1)$ will be a new turning point after a shift.

3.1 Given the equations $h(x) = \tan x$ and $k(x) = -\sin x$, $x \in [0^0;180^0]$



3.2 For which values of x is k undefined?

$x = 90°$

3.3 Write down the period of k

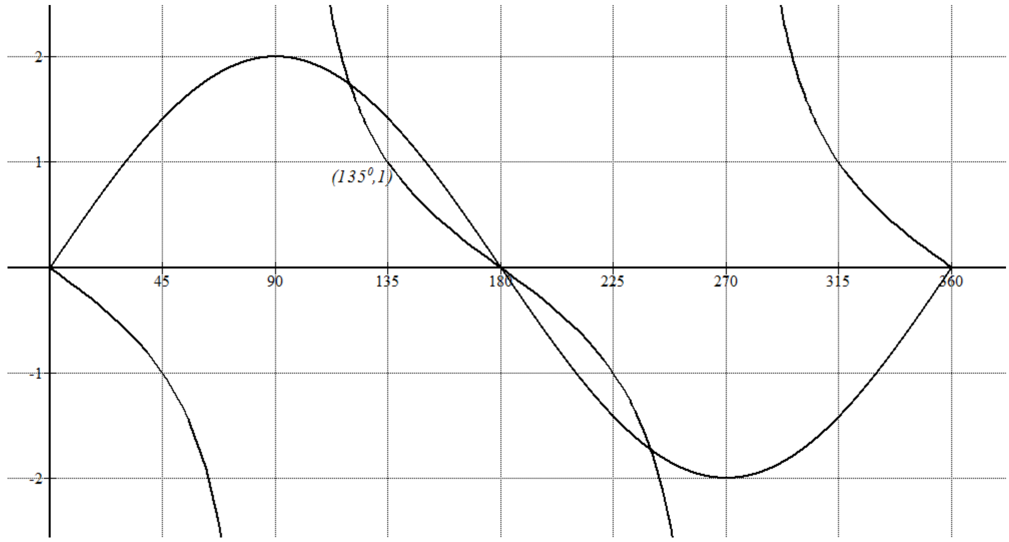
180°

3.4 Write down the maximum value of g if $g(x) = k(x) + 1$, within the given interval.

1

PRACTICE QUESTIONS

1 Given the sketches of $f(x) = a\sin x$ and $g(x) = a\tan x$

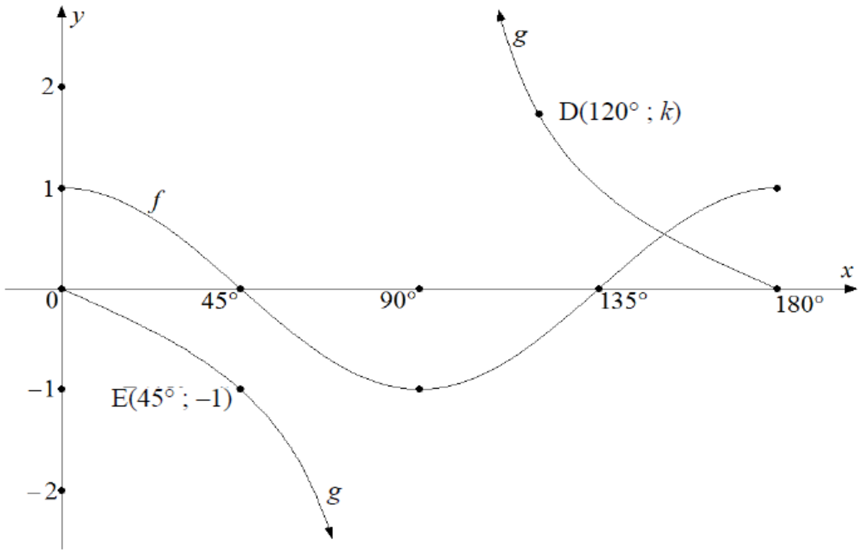


1.1	If $\tan 135° = 1$, then determine the value of a in g.	(1)
1.2	Determine the amplitude of f.	(1)
1.3	Give the period of f.	(1)
1.4	For which values of x will $f(x).g(x) < 0$	(2)
1.5	Write down the equation of the asymptote of g.	(2)

2 Given $f(x) = -2\sin x + 1$ and $g(x) = \cos(x + 30°)$; $x \in [0°; 180°]$

2.1	Draw the two functions in one set of axes. Clearly indicate the coordinates of the turning points and the intercepts with the axes.	(8)
2.2	Write down the range of f.	(2)
2.3	Write down the amplitude of f.	(1)
2.4	Write down the period of g.	(1)
2.5	Indicate on the graph points P and Q, where $-2\sin x = \cos(x + 30°) - 1$	(2)
2.6	For which value(s) of x is $g(x) = 0$	(3)

3 The graph below represents the curves of functions f and g defined by $f(x) = a\cos bx$ and $g(x) = c\tan x$, for $x \in [0°;180°]$. Point D (120^0; *k*) and point E(45^0 ;-1) lie on g.



Use the graph to answer the following

3.1	Give the period of f.	(1)
3.2	Determine the numerical values of *a*, *b*, and *c*.	(4)
3.3	Write down the values of *x* for which $f(x) - g(x) = 1$	()
3.4	Give the equation of the asymptote of g.	(1)
3.5	Determine the numerical value of *k*.	(2)
3.6	Determine the values of *x* for which for $x \in [0°;180°]$ $f(x) \cdot g(x) \leq 0$	(4)
3.7	For which values of *x* will $f'(x) < 0$ (Note f' represents the gradient of a function)?	(2)

SOLUTIONS TO SELECTED QUESTIONS

	TRIG EQUATIONS		
	1	$\cos x = -0{,}349 \; ; 0° \leq x \leq 360°$ $\text{ref angle} = 69{,}6°$ $x = 180° - 69{,}6° = 110{,}4°$ or $x = 180° + 69{,}6° = 249{,}6°$	(3)
	4	tan 2x = 2,114; 2$x \in$ [0 ; 180] $\text{ref angle} = 64{,}68°$ $2x = 64{,}68°$ $x = 32{,}34$ or $2x = 180° - 64{,}68°$ $x = 57{,}66°$	(3)

1	TRIG EXPRESSIONS		
	1.1	$\cos x . \tan^2 x + \cos x$ $= \cos x . \dfrac{\sin^2 x}{\cos^2 x} + \cos x$ $= \dfrac{\sin^2 x}{\cos x} + \cos x$ $= \dfrac{\sin^2 + \cos^2 x}{\cos x}$ $= \dfrac{1}{\cos x}$ or $\sec x$	(3)

	1.2	$\dfrac{\cos^2 x}{1-\sin x}-\sin x$ $=\dfrac{\cos^2 x-\sin x+\sin^2 x}{1-\sin x}$ $=\dfrac{1-\sin x}{1-\sin x}$ $=1$	(4)

2		**TRIG IDENTITIES**	
	2.1	$3\cot^2 x\ (\tan^2 x+1)=3\operatorname{cosec}^2 x$ $\text{LHS}=3\cot^2 x(\sec^2 x)$ $=\dfrac{3\cos^2 x}{\sin^2 x}\times\dfrac{1}{\cos^2 x}$ $=\dfrac{3}{\sin^2 x}=3\cos ec^2 x$	(4)
	2.2	$\dfrac{\sin x}{1-\sin x}+\dfrac{\sin x}{1+\sin x}=\dfrac{2\tan x}{\cos x}$ $\text{LHS}\ =\dfrac{\sin x(1+\sin x)+\sin x(1-\sin x)}{(1-\sin^2 x)}$ $=\dfrac{\sin x+\sin^2 x+\sin x-\sin^2 x}{(1-\sin^2 x)}$ $=\dfrac{2\sin x}{\cos^2 x}$	(4)

		TRIG GRAPHS	
	1.1	$g = 1$	(1)
	1.2	Amplitude = 1	(1)
	1.3	Period $= 360°$	(1)
	1.4	$(0° \; ; \; 90°)$ or $(270° \; ; \; 270°)$	(2)
	1.5	$x = 90°$ or $x = 270°$	(2)
3	**3.1**	Period = $180°$	(1)
	3.2	$a = 1 \; ; \; b = 2 \; ; \; c = -1$	(4)
	3.3	$x = 0° \; ; \; 45° \; ; \; 180°$	(3)
	3.4	$x = 90°$	(1)
	3.5	$k = \sqrt{3}$	(2)
	3.6	$[0°; \; 45°]$ or $[90°; 135°]$	(4)
	3.7	(0º ; 90º)	(2)

2, EUCLIDEAN GEOMETRY

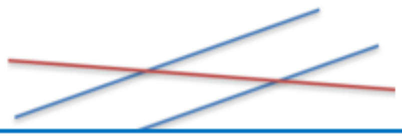
QUESTION P2 **40 MARKS**

OBJECTIVES: After working through this guide you need to be able to do the following:

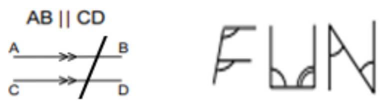
- Understand Geometric terminology for lines and parallel lines, angles, triangle congruency and similarity.
- Apply the properties of line segments joining the mid-points of two sides of a triangle.
- Know the features of the following special quadrilaterals: the kite, parallelogram, rectangle, rhombus, square and trapezium (apply to practical problems).
- Know and apply all the circle theorems.
- Know and apply theorems on similarity and proportionality.

GEOMETRY OF LINES AND ANGLES

TRANSVERSAL: Line passing through two lines

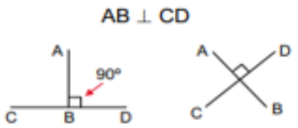


PARALLEL LINES



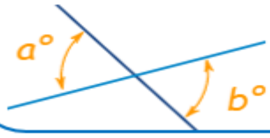
- 'F'.Corresponding ∠s
- 'Z' alternate ∠s
- 'U' co-interior ∠s

PERPENDICULAR LINES



VERTICAL OPPOSITE ANGLES

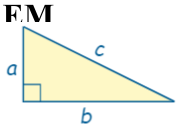
- **When 2 lines intersect**
- **Look for the ' X ' shape**



ANGLES

PYTHAGORUS THEOREM

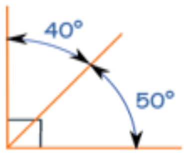
"In a right-angled ***triangle***, ***square*** of the ***hypotenuse side*** is equal to the sum of ***squares*** of the other two ***sides***



$c^2 = a^2 + b^2$

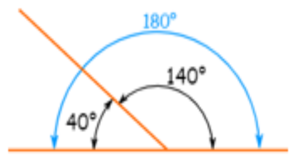
COMPLEMENTARY ANGLES

2 angles add up to 90°



SUBLEMENTARY ANGLES

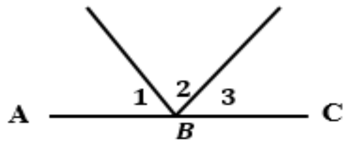
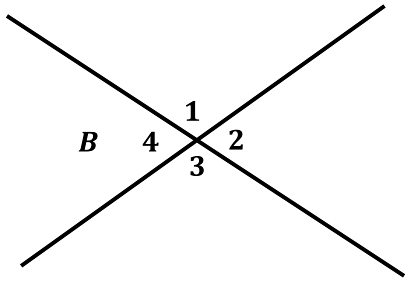
2 angles add up to 180°



THEOREMS ON TRIANGLES	
THEOREM STATEMENT/CONVERSE	**ACCEPTABLE REASON(S)**
The interior angles of a triangle are supplementary.	$\angle$ sum in Δ **OR** sum of $\angle$s in Δ **OR** Int $\angle$s Δ
The exterior angle of a triangle is equal to the sum of the interior opposite angles.	ext $\angle$ of Δ
The angles opposite the equal sides in an isosceles triangle are equal.	$\angle$s opp equal sides
The sides opposite the equal angles in an isosceles triangle are equal.	sides opp equal $\angle$s
In a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.	Pythagoras **OR** Theorem of Pythagoras
If the square of the longest side in a triangle is equal to the sum of the squares of the other two sides then the triangle is right-angled.	Converse Pythagoras **OR** Converse Theorem of Pythagoras
If three sides of one triangle are respectively equal to three sides of another triangle, the triangles are congruent.	SSS
If two sides and an included angle of one triangle are respectively equal to two sides and an included angle of another triangle, the triangles are congruent.	SAS **OR** S$\angle$S
If two angles and one side of one triangle are respectively equal to two angles and the corresponding side in another triangle, the triangles are congruent.	AAS **OR** $\angle\angle$S
The line segment joining the midpoints of two sides of a triangle is parallel to the third side and equal to half the length of the third side	Midpt Theorem
The line drawn from the midpoint of one side of a triangle, parallel to another side, bisects the third side.	line through midpt || to 2^{nd} side

ACTIVITY 1

1. Complete the following statements and give the acceptable reasons

1.1 $\hat{B}_1+\hat{B}_2+\hat{B}_3=$.................		1.1.................................
1.2 $\hat{B}_1+\hat{B}_3+\hat{B}_2+\hat{B}_4=$....... **1.3** $\hat{B}_1+\hat{B}_2$ = ... **1.4** $\hat{B}_3+\hat{B}_4=$.............. **1.5** $\hat{B}_1$ = ... **1.6** $\hat{B}_3=$..........................		1.2 .... 1.3 .... 1.4 .... 1.5 .... 1.6 ....

1.7 Corresponding angles [CD || EF]

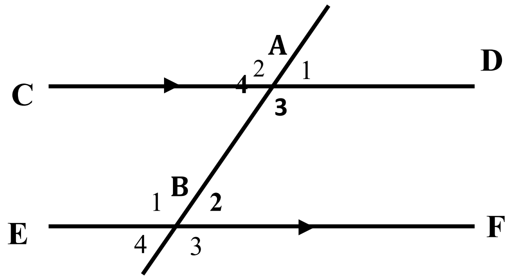
$\hat{A}_1=$...... ; $\hat{A}_2=$.....; $\hat{A}_3=$..... and $\hat{A}_4=$.....

1.8 Co-interior [CD || EF]

$\hat{A}_4+$......$=180°$ and ....+ $\hat{B}_2=180°$

1.9 Alternate angles [CD||EF]

$\hat{A}_4$ = ... and $\hat{B}_1$....... =



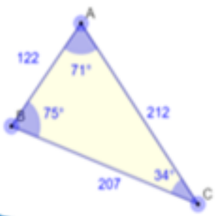
GEOMETRY OF TRIANGLES

TRIANGLES

Angles add always up to 180°

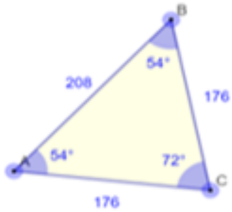
ACUTE TRIANGLE

- All angles are less than 90°



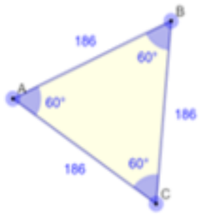
ISOSCELES TRIANGLE

- two sides equal two angles equal



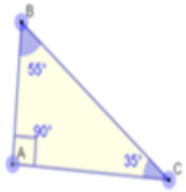
EQUILATERAL TRIANGLE

- Three sides equal three angles equal 60°



RIGHT TRIANGLE

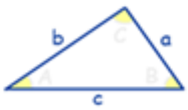
- has a right angle



CONDITIONS FOR CONGRUENCY

(≡) SAME SIZE AND SHAPE

1. **All 3 sides are the same length:**
   (S, S, S)



2. **Two sides and included/joining angle**
   (S, A, S)

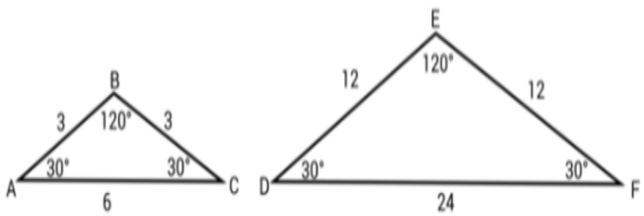
3. **Two angles and a joining side are the same:**
   (A, A, S,)

4. **Right angle, hypotenuse and a side:**
   (90°, H,S)

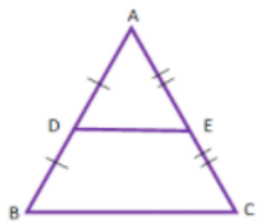
CONDITIONS FOR SIMILARITY

(|||) SAME SHAPE BUT NOT THE SAME SIZE.

1. Two corresponding angles are the same size: (A, A...)
2. Corresponding sides are in proportion equal to each other.



- The **MIDPOINT** is the middle point of a line
- Equidistant from both endpoints
- Centroid of both the segment and the endpoints
- Bisect the segment
- DE = $\frac{1}{2}$ x BC
- DE||BC



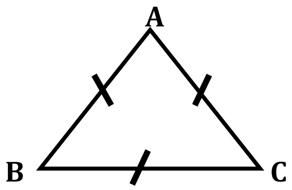
THEOREMS ON LINES AND ANGLES

THEOREM STATEMENT/CONVERSE	ACCEPTABLE REASON(S)
The adjacent angles on a straight line are supplementary.	∠s on a str line
If the adjacent angles are supplementary, the outer arms of these angles form a straight line.	adj ∠s supp
The adjacent angles in a revolution add up to 360°.	∠s round a pt OR ∠s in a rev
Vertically opposite angles are equal.	vert opp ∠s =
If AB ‖ CD, then the alternate angles are equal.	alt ∠s; AB ‖ CD
If AB ‖ CD, then the corresponding angles are equal.	corresp ∠s; AB ‖ CD
If AB ‖ CD, then the co-interior angles are supplementary.	co-int ∠s; AB ‖ CD
If the alternate angles between two lines are equal, then the lines are parallel.	alt ∠s =
If the corresponding angles between two lines are equal, then the lines are parallel.	corresp ∠s =
If the co-interior angles between two lines are supplementary, then the lines are parallel.	co-int ∠s supp

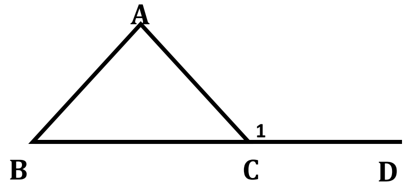
ACTIVITY 2

2.1 Complete the following.

2.1.1 $\hat{A} + \hat{B} =$ ....

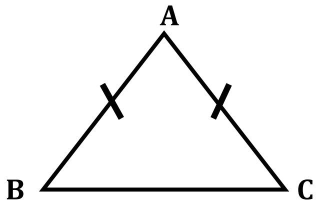


2.1.2 $\hat{A} + \hat{B}$ ....



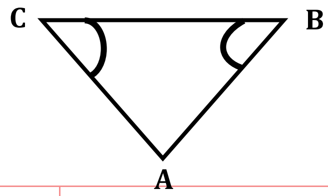
2.1.3 $\angle$s opposite equal sides are equal

If AB = …………, then $\hat{B}$ = …………



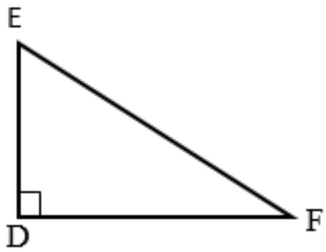
2.1.4 Sides opposite equal $\angle$s

If $\hat{B}$ = ...then .... = AC



2.1.5 $\hat{D}$ ....

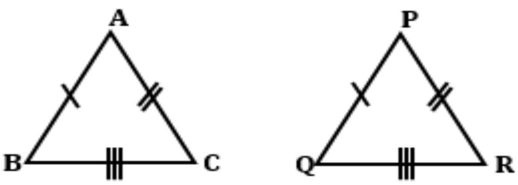
2.1.6 $EF^2 =$ .... + ...



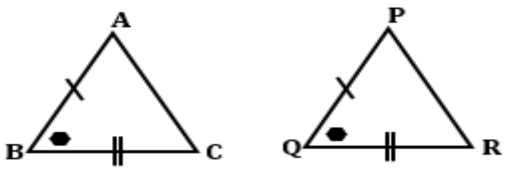
2.1.7 Reason ....

2.2. Give the case of congruency in each of the following:

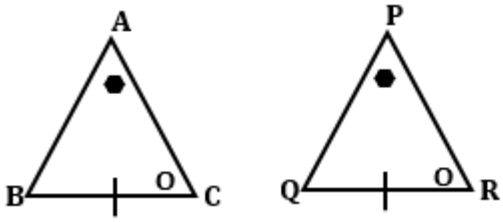
2.2



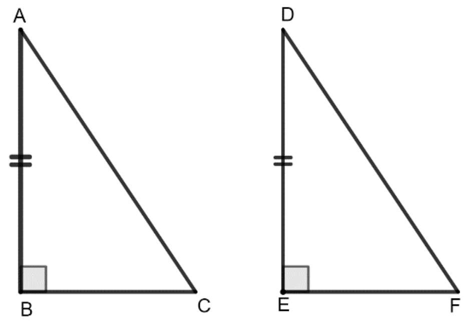
2.2.2 $\Delta ABC \equiv \Delta PQR$ (...........)



2.2.3 $\Delta ABC \equiv \Delta PQR$ (.........)

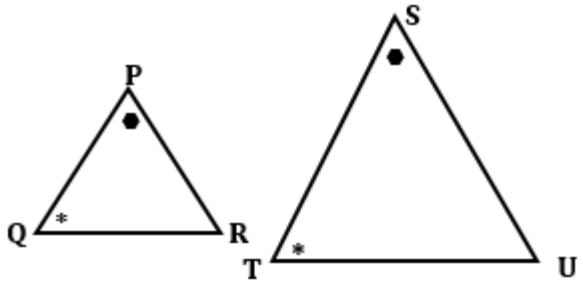


2.2.4 $\Delta ABC \equiv \Delta DEF$ (...........)

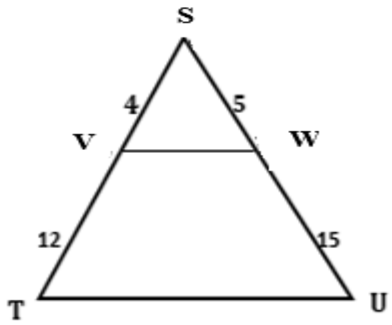


2.3 Give the case of similarity in each of the following.

2.3.1 $\Delta PQR ||| \Delta STU$ ....



2.3.2 $\Delta STU ||| \Delta SVW$ ....

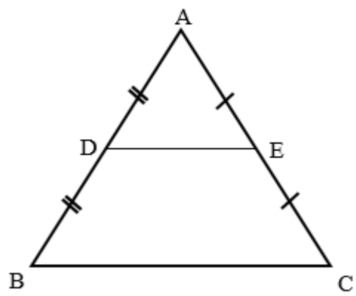


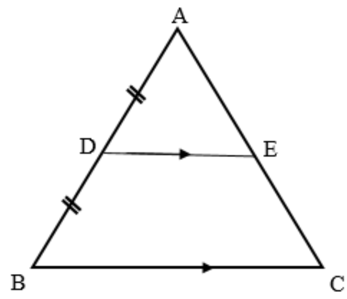
2.4 Complete the following

If AD = DB and AE = EC then,

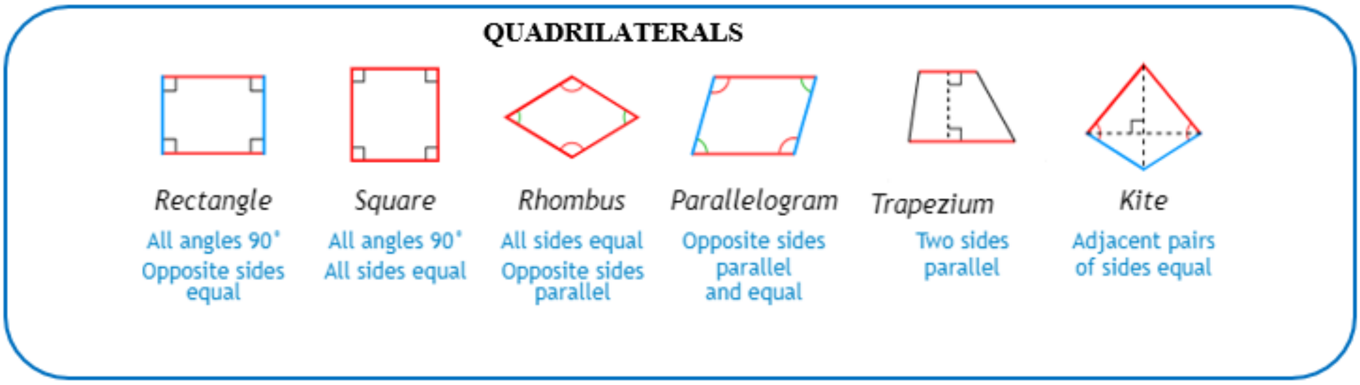
2.4.1 .... || BC and

2.4.2 DE = ...

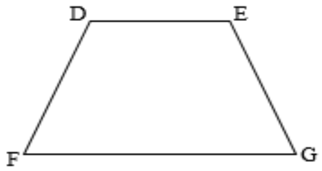
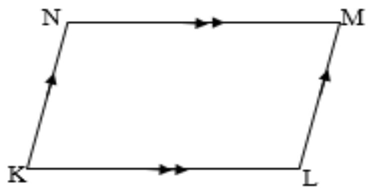
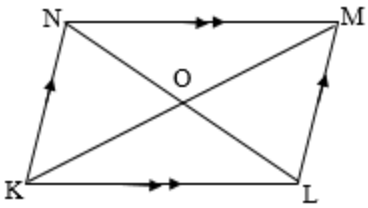
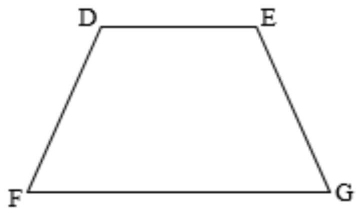


If AD = DB and DE || BC, then **2.4.3** AE = ……. and **2.4.4** ……. $= \frac{1}{2}$ BC	

GEOMETRY OF QUADRILATERALS



The interior angles of a quadrilateral add up to 360°.	sum of $\angle$s in quad
The opposite sides of a parallelogram are parallel.	opp sides of ||m
If the opposite sides of a quadrilateral are parallel, then the quadrilateral is a parallelogram.	opp sides of quad are ||
The opposite sides of a parallelogram are equal in length.	opp sides of ||m
If the opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram.	opp sides of quad are = **OR** converse opp sides of a parm
The opposite angles of a parallelogram are equal.	opp $\angle$s of ||m
If the opposite angles of a quadrilateral are equal then the quadrilateral is a parallelogram.	opp $\angle$s of quad are = **OR** converse opp angles of a parm
The diagonals of a parallelogram bisect each other.	diag of ||m
If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.	diags of quad bisect each other **OR** converse diags of a parm
If one pair of opposite sides of a quadrilateral are equal and parallel, then the quadrilateral is a parallelogram.	pair of opp sides = and ||
The diagonals of a parallelogram bisect its area.	diag bisect area of ||m
The diagonals of a rhombus bisect at right angles.	diags of rhombus
The diagonals of a rhombus bisect the interior angles.	diags of rhombus
All four sides of a rhombus are equal in length.	sides of rhombus
All four sides of a square are equal in length.	sides of square
The diagonals of a rectangle are equal in length.	diags of rect
The diagonals of a kite intersect at right-angles.	diags of kite
A diagonal of a kite bisects the other diagonal.	diag of kite
A diagonal of a kite bisects the opposite angles	diag of kite

ACTIVITY 3		
3 Complete the following and give acceptable reasons.		
3.1 DEGF is a quadrilateral $\hat{D}+\hat{E}+\hat{F}+\hat{G}=$..................		3.1 ....
3.2 MNKL is a parallelogram **3.2.1** NM || .... **3.2.2** NK || .... **3.2.3** ML = ... **3.2.4** KL = ... **3.2.5** $\hat{M}=$ .... **3.2.6** $\hat{N}=$		3.2.1 .... 3.2.2 .... 3.2.3 .... 3.2.4 .... 3.2.6 .... 3.2.7 ....
3.3 MNKL is a parallelogram **3.3.1** ON = ... **3.3.2** KO = ...		3.3.1 .... 3.3.2 ....
3.4 DEGF is the trapezium DE ||		3.4 ....

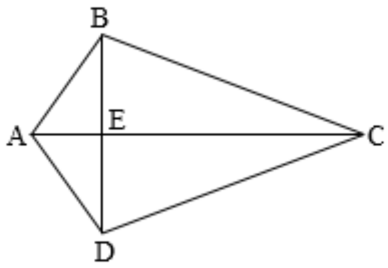
3.5 ABCD is a kite with BD and AC as diagonals

3.5.1 AB = ………

3.5.2 DC = ……...

3.5.3 BE = ……...

3.5.4 $B\hat{E}C$= ……..



3.5.1 ……………………..

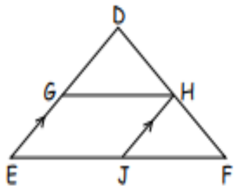
3.5.2 ……………………...

3.5.3 ………………………

3.5.4 ………………………

WORKED EXAMPLES

1. The triangle DEF has G the midpoint of DE, H the midpoint of DF and GH is joined. HJ is parallel to DE



Prove:

1.1 GHJE is a parallelogram (2)

1.2 $D\hat{G}F = D\hat{E}J$ (1)

1.3 JF = GH (3)

1.1 HJ||GE ✓ given

In DEF: G & H are midpoints of DE & D

$\therefore$ GH||EJ and $GH = \frac{1}{2}EF$ ✓ 2 pair opp.sides

$\therefore$ GHJE is a $||^{m}$

1.2 $D\hat{G}F = D\hat{E}J$ ✓corresp ∠s; HJ || DE

1.3 EJ = GH

= $\frac{1}{2}EF$ ✓ Opposite sides of $||^{m}$

JF = $\frac{1}{2}EF$ ✓ $GH = \frac{1}{2}EF$

$\therefore$ JF = GH

$D\hat{G}F = D\hat{E}J$ ✓ answer

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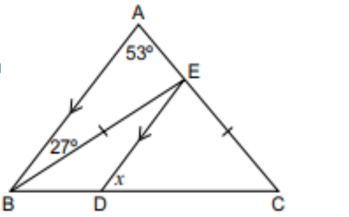
2. In the diagram , AB||ED and BE =E

Also, $A\hat{B}E = 27^\circ$

and $B\hat{A}C = 53^\circ$



2.1 Write down the size of $B\hat{E}D$ and $C\hat{E}D$ with reasons. (2)

2.2 Hence or otherwise, calculate the value of *x*. Show all working and give reasons (4)

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2.1 $B\hat{E}D = 27^\circ$ ✓ alt ∠s =AB||ED

$C\hat{E}D = 53^\circ$ ✓ corresp. ∠s; AB||ED

2.2 $A\hat{B}D = x$ ✓ corresp ∠s; AB||ED

$E\hat{B}C = x - 27^\circ$ ✓∠s opp equal sides

$\hat{C} = x - 27^\circ$

In ΔABC: $53^\circ + x + x - 27^\circ = 180^\circ$ ✓∠ sum in Δ

$2x = 154^\circ$

$x = 77^\circ$ ✓

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3. State whether the following pairs of triangles are congruent or similar, giving reasons for your choice.

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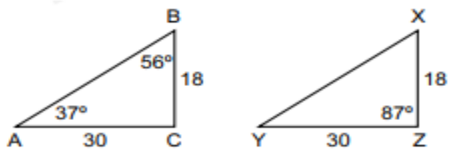
3.1



(2)

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3.1 Congruent ✓ ✓SAS **OR** S∠S

$\hat{C} = 87^\circ$

(2)

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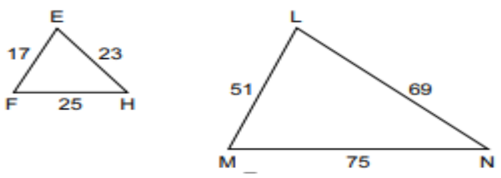
3.2



(2)

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3.2 Similar ✓✓Sides in proportion $\frac{LM}{EF} = \frac{LN}{EH} = \frac{MN}{FH}$; $\frac{51}{17} = \frac{69}{23} = \frac{75}{25} = 3$

(2)

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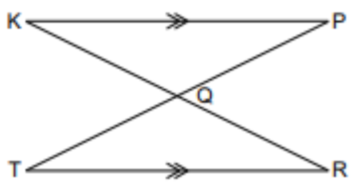
3.3



(3)

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3.3 Similar Equianglar ∠∠∠ ✓Altenate angles KP||TR

$\hat{P} = \hat{T}$ ✓Altenate angles KP||TR

$\hat{K} = \hat{R}$ $K\hat{Q}P = T\hat{Q}R$ ✓ vert. opp ∠s =

(3)

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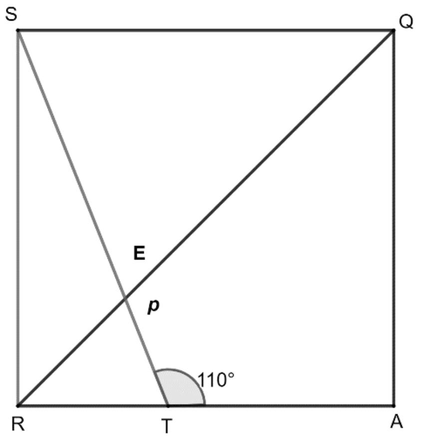
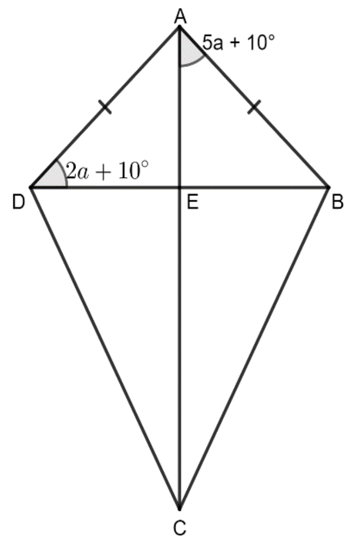
4. Find the value of each of the angles or sides marked with a small letter in the following diagrams. Show all steps and give clear reasons.

4.1 SQAR is a square
(4)



4.2 ABCD is a kite
(3)



SOLUTIONS

4.1 $S\hat{T}R = 180° - 110°$ ✓diags of square

$= 70°$

$R\hat{E}T = 180° - (45° + 70°)$ ✓ sum of □s

$R\hat{E}T = 65°$

$\hat{p} = 180° - 65°$ ✓straight □s

$= 70°$

(4)

4.2 $A\hat{B}E = 5a + 10°$ ✓ diags of kite

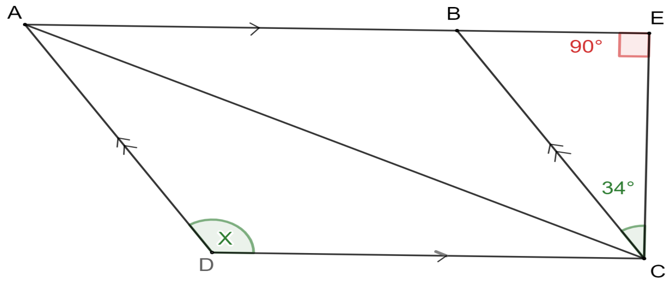
$2a + 10° + 5a + 10° + 90° = 180°$ ✓sum of □s

$7a + 20° = 90°$

$a = 10°$ ✓ answer

(3)

5. Study the diagram below carefully before answering the questions. Quadrilateral ABCD is a parallelogram. BCE is a right angle triangle



5.1 Find the value of $A\hat{B}C$ (2)

5.2 Find the value of *x* (1)

5.3 Prove that ABC and ADC are congruent (3)

5.4 What shape is quadrilateral AECD? (1)

5.5 Is EC parralel to AD? Give a reason for your answer (1)

5.6 Find the value of $D\hat{A}B$ (3)

5.1 $A\hat{B}C = B\hat{E}C + E\hat{C}B$ ✓ Exterior ∠s sum of opposite interior ∠s

$A\hat{B}C = 90^\circ + 34^\circ$

$A\hat{B}C = 124^\circ$ ✓ answer

5.2 $x = A\hat{B}C = 124^\circ$ ✓ opp ∠s of ||m

5.3 In $\Delta A\hat{B}C$ and $A\hat{D}C$

AB = DC ✓ opp sides of ||m

BC = AD ✓ Common

AC =AC

$\therefore \Delta ABC \equiv \Delta DAC$ ✓ SSS

5.4 Trapezium. ✓ Opposite sides parallel, no sides are equal.

5.5 No, because AD ||BC and EC meet at C. Therefore EC cannot be parallel to AD✓

5.6 $D\hat{A}B + A\hat{D}C = 180^\circ$ ✓co-int ∠s; AB || CD

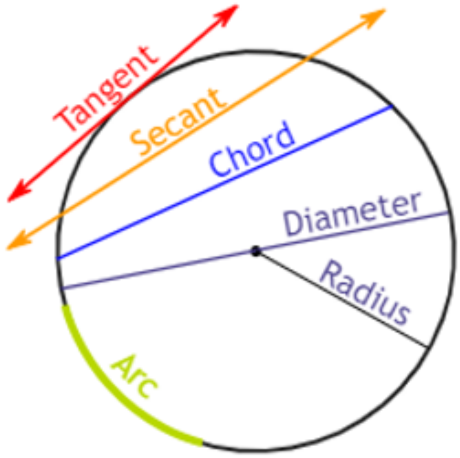
$\therefore D\hat{A}B = 180^\circ - 124^\circ$ ✓proven above

$D\hat{A}B = 56^\circ$ ✓ Answer

GEOMETRY OF CIRCLES

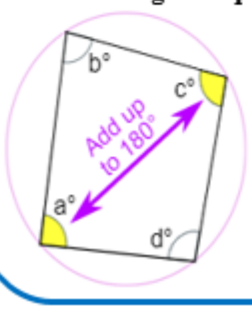
TERMINOLOGY

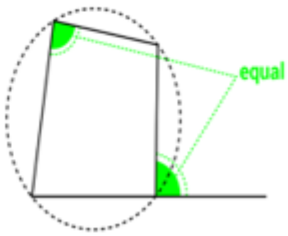
- A **CHORD** is a straight line that joins two points on the circumference of a circle.
- A **DIAMETER** is a chord that passes through the centre of the circle.
- A **RADUIS** is a line joining the centre of the circle to a point on the circumference
- A **TANGENT** is a straight line that touches a circle at one point only and when extended it doesn't cut the circle.
- A **SECANT** is a straight line that cuts a circle in two places.



CYCLIC QUADRILATERALS

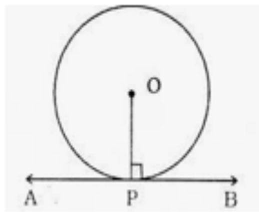
- **A quadrilateral which is circumscribed in a circle**
- **It means that all four vertices of the quadrilateral lie in the circumference of the circle.**
- **Opposite angles add to** 180°
- **Exterior angle is equal to the opposite interior angle**



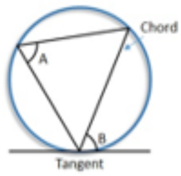


TANGENTS

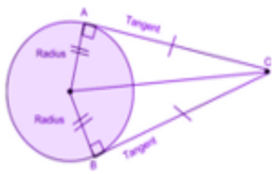
- Tangents always forms a right angle with the circles radius



- The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment.

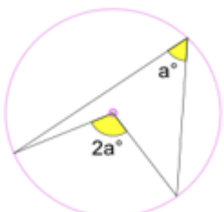


- Two tangents drawn to a circle from the same point outside the circle are equal in length

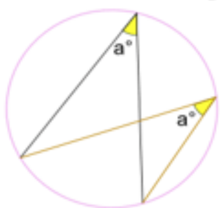


INSCRIBED ANGLES

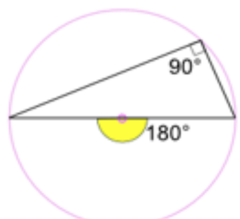
- An inscribed angle a° is half of the centre angle $2a^\circ$



- No matter where it is on arc the angle a° is always the same between end points



- An inscribed angle 90° is half the central angle 180°

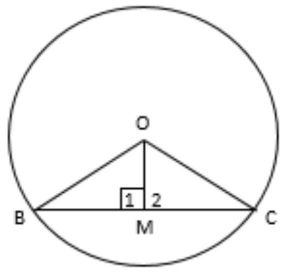
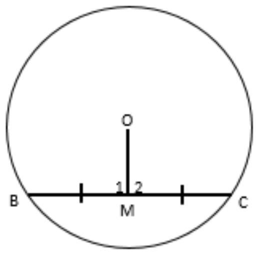


CIRCLE THEOREMS	
The tangent to a circle is perpendicular to the radius/diameter of the circle at the point of contact.	tan∠radius tan∠diameter
If a line is drawn perpendicular to a radius/diameter at the point where the radius/diameter meets the circle, then the line is a tangent to the circle.	line ∠ radius **OR** converse tan ∠ radius **OR** converse tan diameter
The line drawn from the centre of a circle to the midpoint of a chord is perpendicular to the chord.	line from centre to midpt of chord
The line drawn from the centre of a circle perpendicular to a chord bisects the chord.	line from centre ∠ to chord
The perpendicular bisector of a chord passes through the centre of the circle;	Perp. bisector of chord
The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre)	∠ at centre = 2 ×∠ at circumference
The angle subtended by the diameter at the circumference of the circle is 90°.	∠s in semi- circle **OR** diameter subtends right angle
If the angle subtended by a chord at the circumference of the circle is 90°, then the chord is a diameter.	chord subtends 90∠ **OR** converse ∠s in semi -circle
Angles subtended by a chord of the circle, on the same side of the chord, are equal	∠s in the same seg.
If a line segment joining two points subtends equal angles at two points on the same side of the line segment, then the four points are concyclic.	line subtends equal ∠s **OR** converse ∠s in the same seg.
Equal chords subtend equal angles at the circumference of the circle.	equal chords; equal ∠s
Equal chords subtend equal angles at the centre of the circle.	equal chords; equal ∠s
Equal chords in equal circles subtend equal angles at the circumference of the circles.	equal circles; equal chords; equal ∠s
Equal chords in equal circles subtend equal angles at the centre of the circles.	equal circles; equal chords; equal ∠s
The opposite angles of a cyclic quadrilateral are supplementary	opp ∠s of cyclic quad
If the opposite angles of a quadrilateral are supplementary then the quadrilateral is cyclic.	opp ∠s quad supp **OR** converse opp ∠s of cyclic quad
The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.	ext ∠ of cyclic quad
If the exterior angle of a quadrilateral is equal to the interior opposite angle of the quadrilateral, then the quadrilateral is cyclic	ext ∠ = int opp ∠ **OR** converse ext ∠ of cyclic quad

Two tangents drawn to a circle from the same point outside the circle are equal in length	Tans from common pt **OR** Tans from same pt
The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment.	tan chord theorem
If a line is drawn through the end-point of a chord, making with the chord an angle equal to an angle in the alternate segment, then the line is a tangent to the circle.	converse tan chord theorem **OR** ∠ between line and chord

ACTIVITY 4

Complete the following

4.1 Circle centre O with chord BC. OM⊥BC. **4.1.1** Δ OBC is …………………….. triangle[kind of triangle] **4.1.2** Δ OBM and Δ OCM are ………… triangles [kind of triangle**]**		**ACCEPTABLE REASON(S)** 4.1.1 ……………………… 4.1.2 ………………………..
4.2 O is the centre of the circle and BM = MC. **4.2.1** OM is ⊥ on ………… **4.2.2** OC = …………………..		4.2.1 ……………………… 4.2.2 ……………………...

4.3 The perpendicular bisector of a chord passes through the centre of the circle.

O is the centre of the circle and

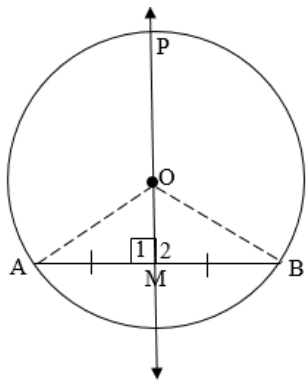
AM = BM and $M_1 = 90^\circ$

4.3.1 $\therefore \Delta\ AOM \equiv \Delta$ ……….. S∠S

4.3.2 $\therefore AO =$ …….. radius

All points on PM is equidistant from A and B.
The centre of the circle is also equidistant from A and B

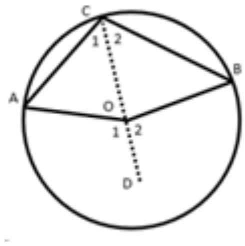
4.3.3 ∴ …………lies on the center of the circle

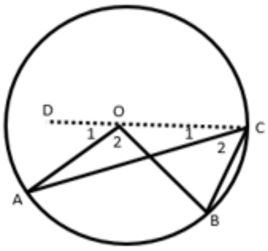


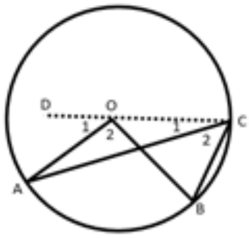
4.4 The angle that an arc of a circle subtends at the centre of the circle is twice the angle it subtends at any point on the circle's circumference.

Circle centre O and arc AB subtending $A\hat{O}B$ at the centre and $A\hat{C}B$ at the circumference.

$A\hat{O}B$= ………………………………..





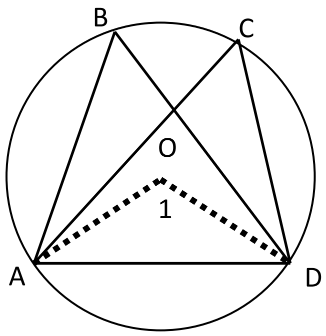


4.5 Angles subtended by a chord (or arc) at the circumference of a circle on the same side of the chord are equal; or angles in the same segment are equal.

Circle centre O and chord AD (or arc) subtended

$A\hat{B}D$and $A\hat{C}D$in the same segment.

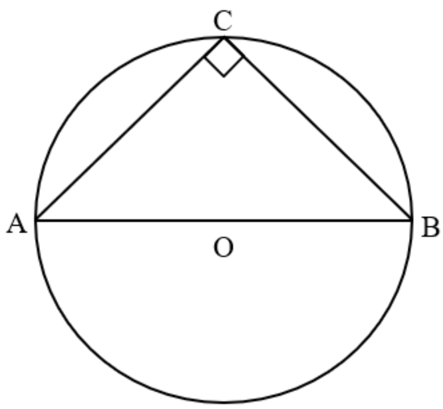
$A\hat{B}D$ = ……………………………………………



4.6 If a chord subtends an angle of 90° at the circumference of a circle, then that chord is a diameter of the circle.

ΔABC with Chord AB and $\hat{C} = 90^0$

AB is a …………………………………………..

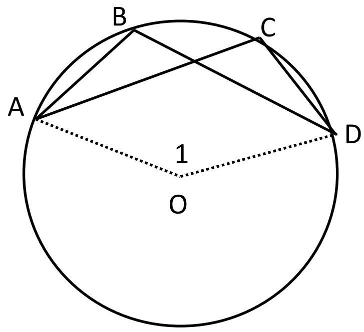


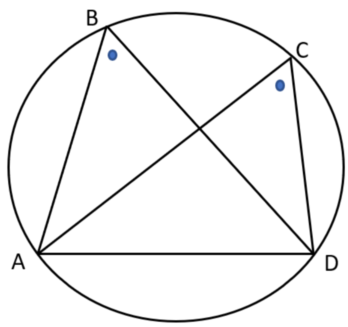
4.7 If a line segment joining two points subtends equal angles at two other points on the same sides of the line segment, then these four points are concylic (that is, they lie on the circumference of a circle).

Given: $\hat{B} = \hat{C}$

Then a circle can be drawn to

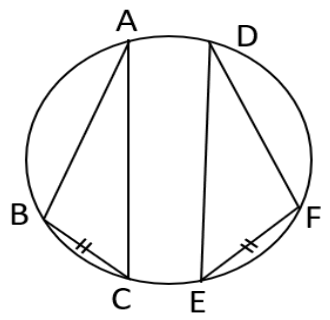
……………………………..





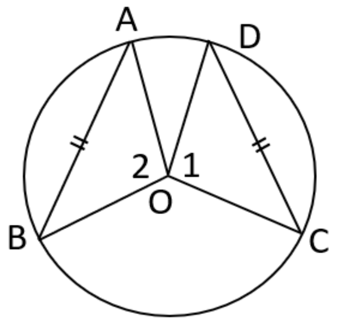
4.8 Equal chords (or arcs) of a circle subtended equal angles at the circumference of a circle.

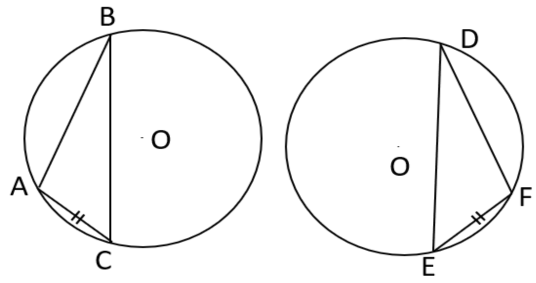
Chord BC = chord EF, then $\hat{A}$ =…………………



4.9 Equal chords (or arcs) of a circle subtend equal angles at the centre of a circle.

AB = DC then $\hat{O}_1$ =…………………………..



4.10 The angles substended **chords (or arcs) of equal length in two different circles with equal radii are equal.** Circle centre O with AC = EF and radii of the circles equal. $\hat{B} = ...$	

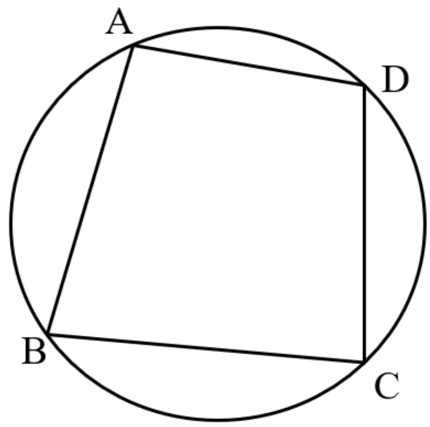
ACTIVITY 5

Use the statements to complete the following.

5.1 If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

Given: $\hat{A}+\hat{C}=180^\circ$ or $\hat{B}+\hat{D}=180^\circ$

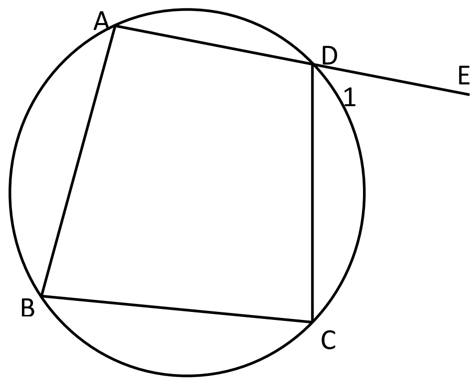
Then ABCD is a …………………….. That is, a circle can be drawn to pass ………………………………



5.2 An exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

ABCD is a cyclic quad and AD is produced to E to form exterior angle $\hat{D}$,

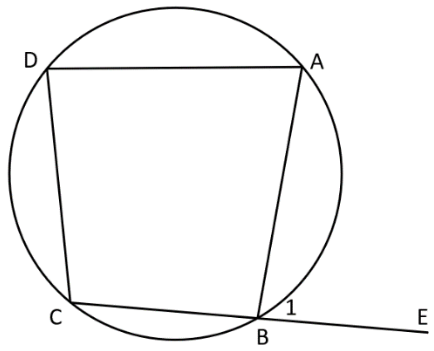
Then $\hat{D}_1$……………………………..



5.3 If an exterior angle of a quadrilateral is equal to the interior opposite angle, then the quadrilateral is cyclic.

Quadrilateral ABCD with CB extended to E and $\hat{B}_1 = \hat{D}$

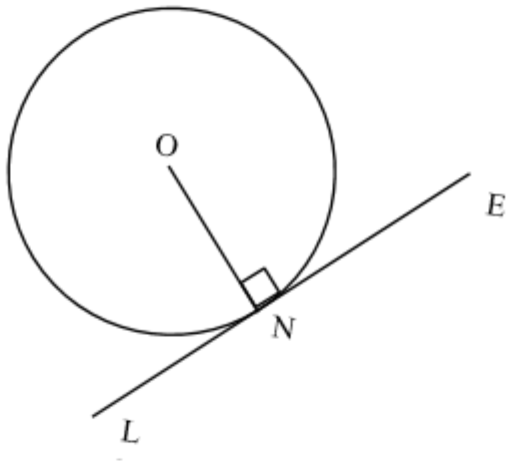
Then ABCD is a ………………………..…………



5.4 **A line drawn perpendicular to a radius at the point where the radius meets the circle is a tangent to the circle**

ON is a radius and perpendicular line LNE at N

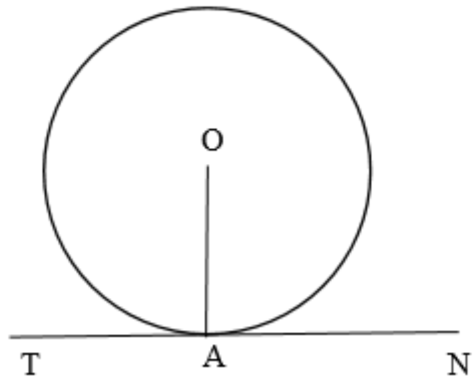
Then: LNE is a …………………………

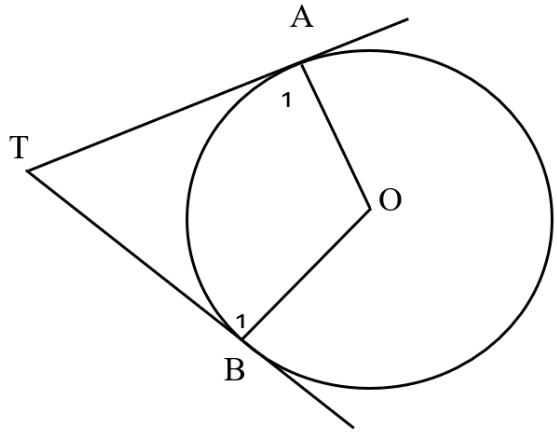
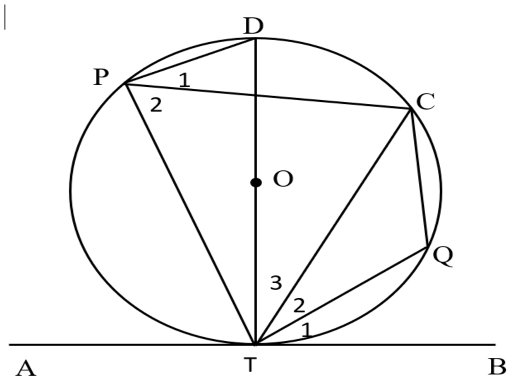


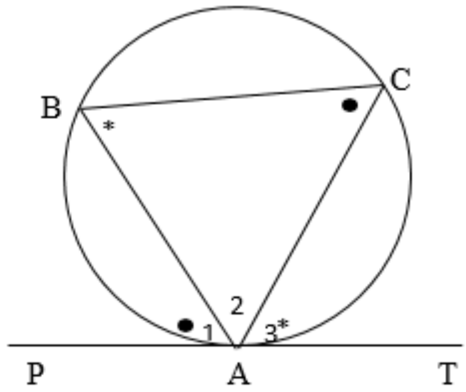
5.5 **A tangent to a circle is perpendicular to the radius at its point of contact.**

TAN is a tangent to the circle centre O at A. OA is a radius

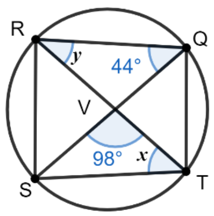
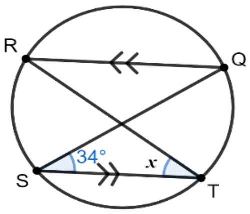
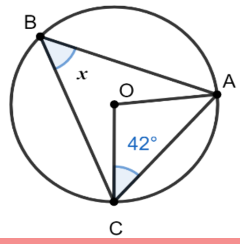
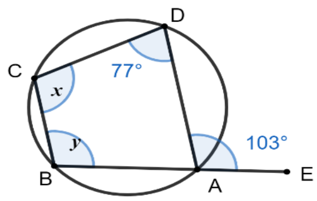
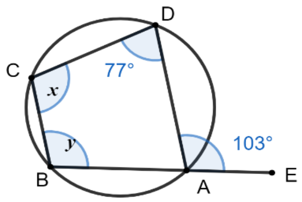
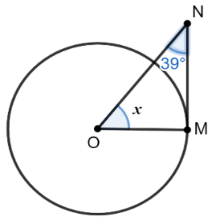
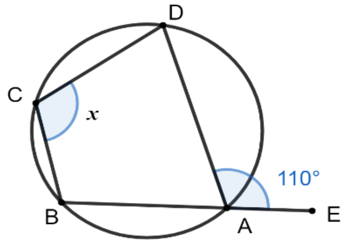
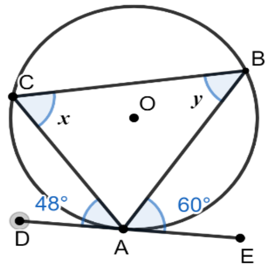
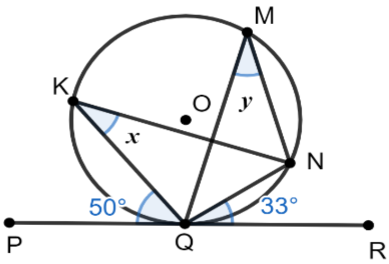
Then OA ⊥ ………………………..



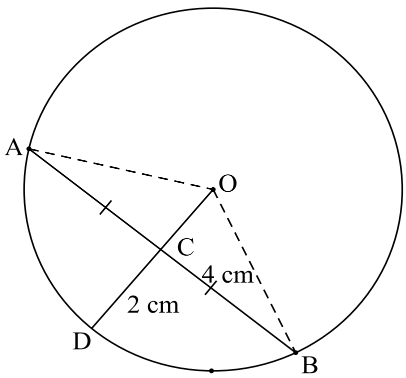
5.6 Two tangents drawn to a circle from the same point outside the circle are equal in length. Circle centre O and tangents TA and TB touching the circle at A and B respectively. Then TA = ………..	
5.7 The angle between a tangent to a circle and a chord drawn from the point of contact is equal to an angle in the alternate segment. Circle centre O with tangent ATB at T, and P, D, C and Q are points on the circle **Then** 5.7.1 $\hat{T}_1 + \hat{T}_2 =$ ………………………………. 5.7.2 $A\hat{T}C =$ ………………………………..	

5.8 If a line is drawn through the end point of a chord, making an angle equal to an angle in the alternate segment, then the line is a tangent to the circle. If $\hat{A}_3 = \hat{B}$ or if $\hat{A}_1 = \hat{C}$ **Then:** PAT is ……………………….at A	

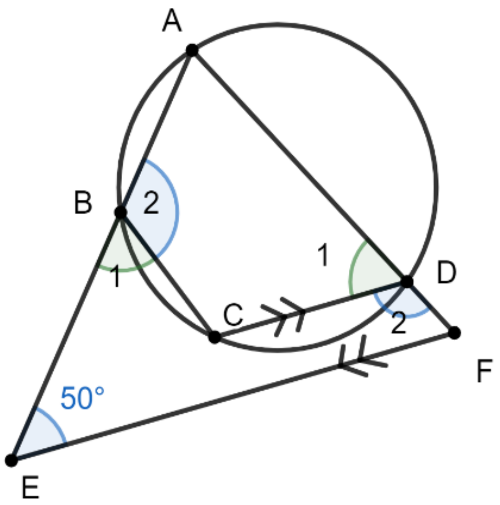
WORKED EXAMPLES

1. Calculate the values of the unknown angles. O is the centre of the circle.		
1.1 	**1.2** 	**1.3** 
SOLUTIONS		
$x = 44^\circ$ ✓□s in the same seg. $\hat{V}_1 = 98^\circ$ ✓ vertical opposite $y = 180^\circ - (98^\circ + 44^\circ)$ $y = 38^\circ$ ✓ answer (3)	$\hat{Q} = 34^\circ$ ✓ alternate □s RQ||ST $x = 34^\circ$ ✓ □s in the same seg (2)	$C\hat{A}O = 42^\circ$ ✓✓radii $C\hat{O}A = 180^\circ - (42^\circ + 42^\circ)$ □ sum in □ $= 96^\circ$ ✓ $x = 48^\circ$ ✓□ at centre = 2 ×□ at circumf. (4)
1.4 	**1.5** 	**1.6** 
SOLUTIONS		
$x = 130^\circ$ ✓ext □ of cyclic quad $y = 103^\circ$ ✓opp □s of cyclic quad	$x = 110^\circ$ ✓ ext □ of cyclic quad $y = 115^\circ$ ✓ ext □ of cyclic quad	$\hat{M} = 90^\circ$ ✓tang ⊥ raduis $x = 180^\circ - (90^\circ + 35^\circ)$ □ sum in □ $x = 55^\circ$ ✓
1.7 	**1.8** 	**1.9** 

SOLUTIONS		
$x = 110^\circ$ ✓ext □ of cyclic quad (1)	$x = 64^\circ$ ✓tan-chord theorem $y = 48^\circ$ ✓tan-chord theorem (2)	$x = 33^\circ$ ✓tan-chord theorem $y = 33^\circ$ ✓∠s in the same seg (2)

1 AB = 8 cm is the chord of the circle with centre O. OCD is the radius of the circle with C on AB such that C is the midpoint of AB. If DC = 2 cm, Calculate the radius of the circle  .	**1** CB = 4 cm — C is the midpoint of AB OC ⊥ AB — Line from centre to midpoint of chord AB $OC^2 + CB^2 = OB^2$ — Pythagoras But OC = OB – 2 — Radii $(OB - 2)^2 + 4^2 = OB^2$ $OB^2 - 4OB + 4 + 16 = OB^2$ $-4OB = -20$ OB = 20 cm

2 In the diagram below, ABCD is a cyclic quadrilateral with AD produced to F and AB produced to E. CD||EF $\hat{E} = 50^\circ$ and EA = AF
.



2.1 Caculate $\hat{B}_2$

2.2 $\hat{B}_1$

2.1 In ΔAEF

EA = AF — Given

$\hat{F} = \hat{E} = 50^\circ$ — Corr. ∠s CD||EF

$\hat{F} = \hat{D}_1 = 50^\circ$ — ∠s opp. equal sides

$\hat{D}_1 + \hat{B}_2 = 180^\circ$ — Opp. ∠s of cyclic quad

$50^\circ + \hat{B}_2 = 180^\circ$

$\hat{B}_2 = 130^\circ$

2.2 $\hat{B}_1 = \hat{D}_1$ — Ext. ∠ of Δ

$= 50^\circ$

OR

$\hat{B}_1 + B_2 = ...$ Sum of ∠s of Δ

$\hat{B}_1 = 180^\circ - 130^\circ$
$= 50^\circ$

3 In the diagram below, A, B, C and D are points on a circle having centre O. PBT is a tangent to the circle at B.

Reflex $B\hat{O}C = \hat{O}_1 = 310^\circ$ as shown in the diagram below.

3.1 $\hat{O}_2 = 50^\circ$ — ∠*s* around a point

$\hat{D}_1 = 25^\circ$ — ∠centre = 2× ∠ at circumference

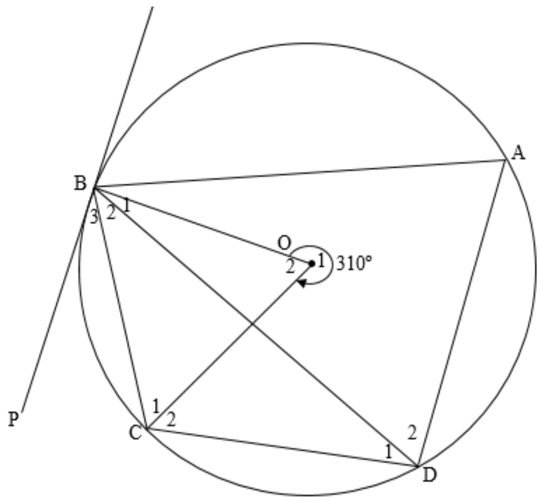
3.2 $\hat{B}_3 = 25^\circ$ — tan chord theorem

3.3. $B\hat{C}D = 180^\circ - 60^\circ$ — Opp. ∠s of cyclic quad

$\hat{B}_2 = 35^\circ$

$\hat{B}_2 = 35^\circ$

$O\hat{B}C = O\hat{C}B = 65^\circ$



3.1 $\hat{D}_1$ (2)

3.2 $\hat{B}_3$ (2)

3.3 $\hat{B}_1$, if it is given that $\hat{A} = 60^{\circ}$. (4)

$\therefore \hat{B}_1 = 65^{\circ} - 35^{\circ}$

$\hat{B}_1 = 30^{\circ}$

Opp. ∠s of cyclic quad

OR

$B\hat{C}D = 120^{\circ}$

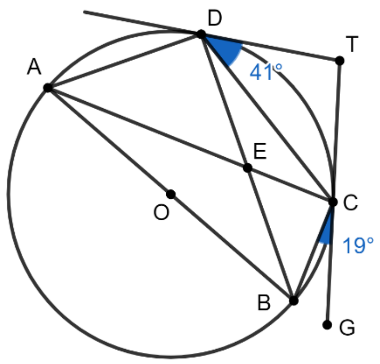
$\hat{B}_2 = 35^{\circ}$

$\hat{B}_1 + \hat{B}_2 + \hat{B}_3 = 90^{\circ}$

raduis ⊥ tangent

$\hat{B}_1 = 30^{\circ}$

4 In the diagram below, ABCD is a cyclic quadrilateral with AB the diameter of the circle. DT and TG are tangents to the circle with and $B\hat{C}G = 19^{\circ}$ AC and BD are drawn to intersect at E.



4.1 Name with reasons THREE other angles equal to . (5)

4.2 Determine, with reasons, the size of $A\hat{B}E$ (4)

4.1 TD = TG Tan from the same point

$T\hat{D}C = T\hat{C}D$ ∠s opp. equal sides

$= 41^{\circ}$

$D\hat{A}C = T\hat{D}C$ tan-chord theorem

$= 41^{\circ}$

$D\hat{A}C = C\hat{B}D$ tan-chord theorem

$= 41^{\circ}$

4.2 $A\hat{C}B = 90^{\circ}$ raduis $\perp$ tangent

$B\hat{A}C = 19^{\circ}$ tan-chord theorem

$B\hat{A}C = 19 A\hat{B}C + B\hat{A}C + C\hat{B}D = 180^{\circ}$

$A\hat{B}E = 180^{\circ} - 19^{\circ} - 90^{\circ} - 41^{\circ} = 30^{\circ}$ ∠ sum in ∠

GEOMETRY OF SIMILARITY AND PROPORTIONALITY

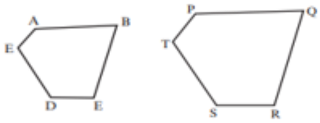
POLYGONS are similar shapes, different in size under the following conditions:

- **All pairs of corresponding angles are equal (equiangular)**
- **All pairs of corresponding sides are in the same proportion**

Corresponding sides are in the same position

Consider pentagon **ABCD** and pentagon **PQRST**

- $\hat{A} = \hat{P}; \hat{B} = \hat{Q}; \hat{C} = \hat{R}; \hat{D} = \hat{S}; \hat{E} = \hat{T}$
- $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{DC}{SR} = \frac{ED}{TS} = \frac{EA}{TP}$ $\therefore ABCD \,|||\, PQRST$



TRIANGLE are special polygons:

- If two triangle are equiangular, then their sides will always be in the same proportion, so the angles are **similar.**
- If the sides of two triangles are in the same proportion, then the triangles will be equiangular, so that the triangles are **similar.**

PROPORTION: two ratios are equal

$2(9) = 3(6)$

$18 = 18$

THEOREMS OF SIMILARITY AND PROPORTIONALITY	
The line drawn from the midpoint of one side of a triangle, parallel to another side, bisects the third side.	line through midpt || to 2^{nd} side
A line drawn parallel to one side of a triangle divides the other two sides proportionally.	line || one side of $\angle$ **OR**
If a line divides two sides of a triangle in the same proportion, then the line is parallel to the third side.	line divides two sides of Δ in prop
If two triangles are equiangular, then the corresponding sides are in proportion (and consequently the triangles are similar).	||| $\angle$s **OR** equiangular Δs
If the corresponding sides of two triangles are proportional, then the triangles are equiangular (and consequently the triangles are similar).	Sides of Δ in prop

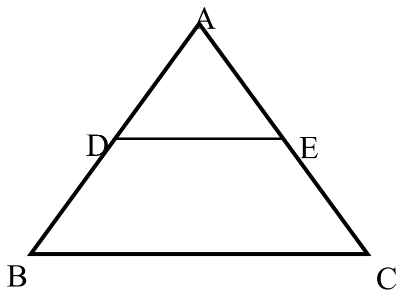
ACTIVITY 6

Complete the following

6.1 If DE ||…..

then $\frac{AD}{DB} = \frac{AE}{EC}$

OR AD : DB = AE : EC



If a line divides two sides of a triangle proportionally, then the line is parallel to the third side of the triangle (line divides two sides of Δ in prop)[name || lines]

If $\frac{AD}{DB} = \frac{AE}{EC}$, **OR** AD : DB = AE : EC

then DE ǁ ………..

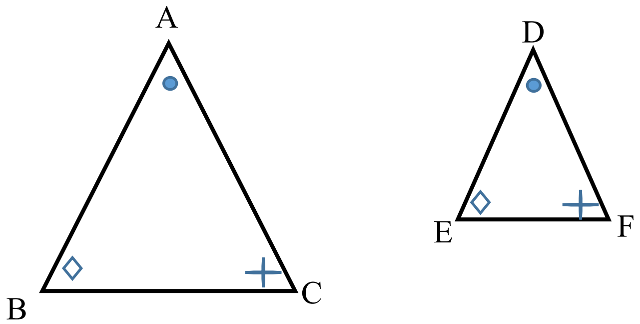
(||| Δs OR equiangular Δs)

The corresponding sides of two equiangular, triangles are proportional and consequently the triangles are similar.

If Δ ABC||| Δ DEF then

NB: Converse; If the sides of two triangles are proportional then the triangles are …….. and consequently the triangles are ………..

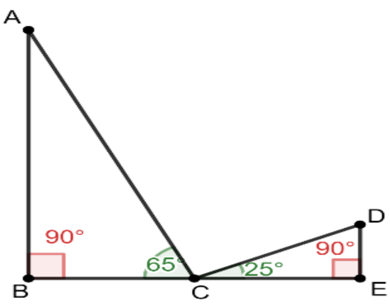
If $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$ then Δ ABC ||| Δ DEF



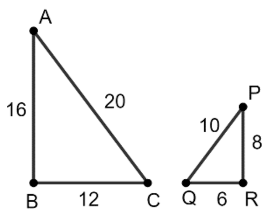
WORKED EXAMPLE

Study the two diagrams below and fill in the correct answer.

1.



2.



SOLUTIONS

1. In ΔABC and ΔDEC

1.1 $\hat{A} = 25^\circ$ and $\hat{D} = ...$ Sum of $\angle$s of Δ

1.2 $\hat{A} = \hat{C} = ...$ given

1.3 $\hat{B} = ... = 90^\circ$

1.4 $\therefore \Delta ABC \,|||\, \Delta DEC$ Reason ……

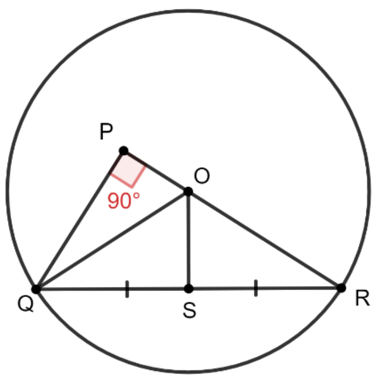
2. In ΔABC and ΔPRQ

2.1 $\frac{AB}{PR} = \frac{......}{RQ} = \frac{AC}{.......}$ Prop. Theorem

2.2 $\frac{16}{.....} = \frac{12}{6} = \frac{20}{.....} = ...$

2.3 $\therefore \Delta ABC \,|||\, \Delta PRQ$ Reason…………

3. In the diagram below:



3.1 Prove $\Delta PRQ \,|||\, \Delta SRO$

3.2 Prove $\frac{OR}{SR} = \frac{QR}{PR}$

3.3 If SR = 18mm and QP=20mm.

Show that the radius of the circle is $\text{raduis} = 21.6\text{mm}$ (correct to 1 decimal place)

3.1 In ΔPRQ and ΔSRO Given

$\hat{P} = 90^\circ$

$\hat{S} = 90^\circ$ QR = SR common

$\hat{P} = \hat{S}$

$\hat{R} = \hat{R}$ A A A

$\therefore \Delta PRQ \,|||\, \Delta SRO$

3.2 $\frac{PR}{SR} = \frac{RQ}{RO} = \frac{PQ}{SO}$ $\therefore \Delta PRQ|||\Delta SRO$

So $\frac{RQ}{RO} = \frac{PR}{SR}$ Prop

QR.SR = PR.OR Cross multiply

$\therefore \frac{OR}{SR} = \frac{QR}{PR}$

	3.3 $\therefore \frac{OR}{SR} = \frac{QR}{PR}$ proved $\Delta QPR: PR^2 = QR^2 - QP^2$ Pythagoras Theorem $= (36)^2 - (20)^2$ $PR = \sqrt{896}$ $= 8\sqrt{14}$ $\frac{OR}{18} = \frac{36}{8\sqrt{14}}$ substitute $\text{radius} = 21{,}6\,\text{mm}$

CONSOLIDATION EXERCISE

<table>
<tr><td>

1. Use the following cords to write down all the inscribed angles equal to each other. (Angle in the same segment)

1.1 ED: ……………………..
(2)

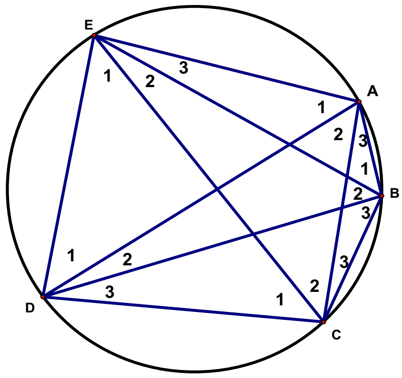
1.2 AE: ……………………..
(2)

1.3 AB: ……………………..
(2)

1.4 BC: …………………....
(2)

1.5 EB: ……………………..
(2)

</td><td>



</td></tr>
<tr><td>

2. In the diagram below, AOD is a diameter of the circle with centre O. BC = CD and

Determine, with reasons, the size of each of the following angles:

2.1 $\hat{A}_2$ (2)

2.2 $\hat{D}$ (3)

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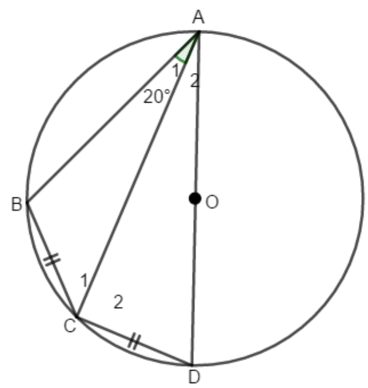


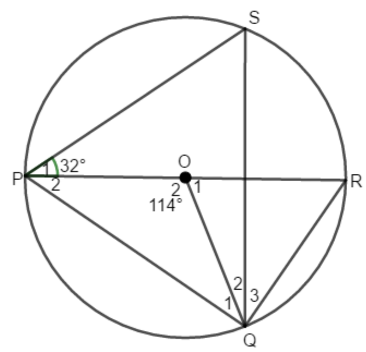
</td></tr>
</table>

3. In the diagram below, a circle with centre O passes through P, Q, R and S. PR is a diameter of the circle. $P\hat{Q}R = 114^\circ$ and $S\hat{P}R = 32^\circ$ Determine, with reasons, the size of the following angles:

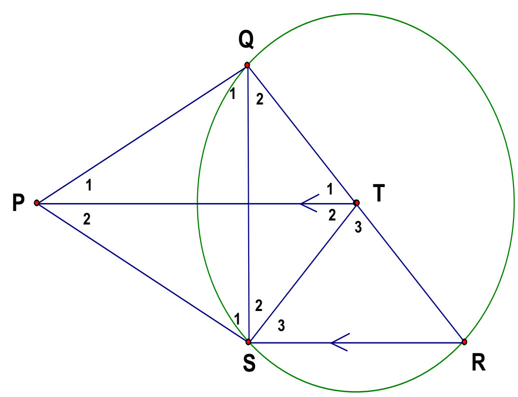
3.1 $P\hat{S}Q$ (2)

3.2 $\hat{Q}_3$ (2)

3.3 $P\hat{Q}S$ (3)



4. In the diagram, circle QRS has tangents PQ and PS. PT || SR with T on QR.



4.1 Find, with reasons, 3 other angles equals to *x*.

4.2 Show that:

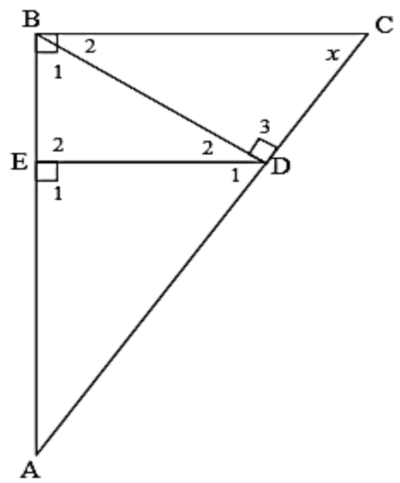
4.2.1 TQPS is a cyclic quadrilateral. (3)

4.2.2 $\hat{S}_3 = \hat{Q}_1$ (3)

5 ΔABC is a right-angled triangle with $S\hat{P}R = 32^{\circ}$.
D is a point on AC such that BD ⊥ AC and
E is a point on AB such that DE ⊥ AB. E and D are joined.
AD : DC = 3 : 2.
AD = 15 cm.

5.1 Calculate BD
(Leave your answer in surd form) (4)

5.2 Show that AE = $AE = 3\sqrt{15}$ (5)

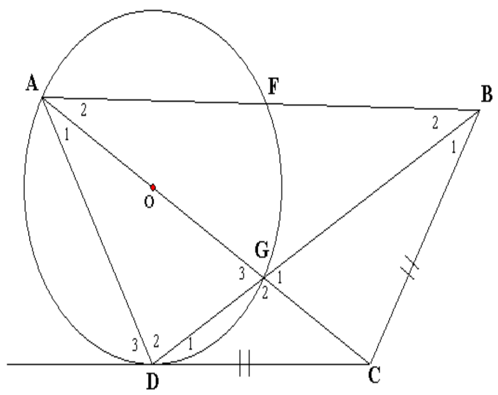


6 In the figure below, AG the diameter of the circle O is produced to C. DC is the tangent to the circle at D. A, D, G and F lie on the circumference of the circle and DC = BC. AFB is a straight line and $\hat{A}_1 = 21^{\circ}$.
Determine, giving reasons, the sizes of the following angles:

6.1 $\hat{D}_1$ (2)

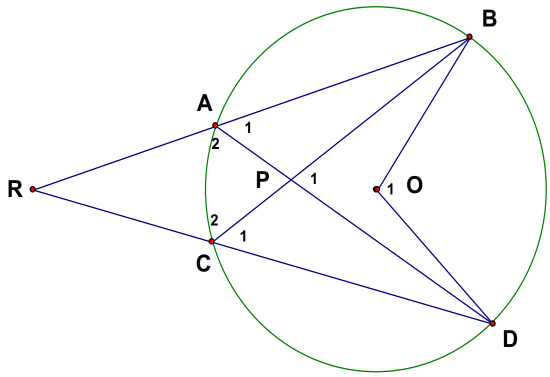
6.2 B_1 (2)

6.3 $\hat{G}_2$ (4)



7 Two secants, RAB and RCD of a circle O intersect the circle at A, B, C and D respectively. AD and BD intersect in P. BO and DO are joined.

Prove that: $\hat{O}_1 = \hat{P}_1$ (2)

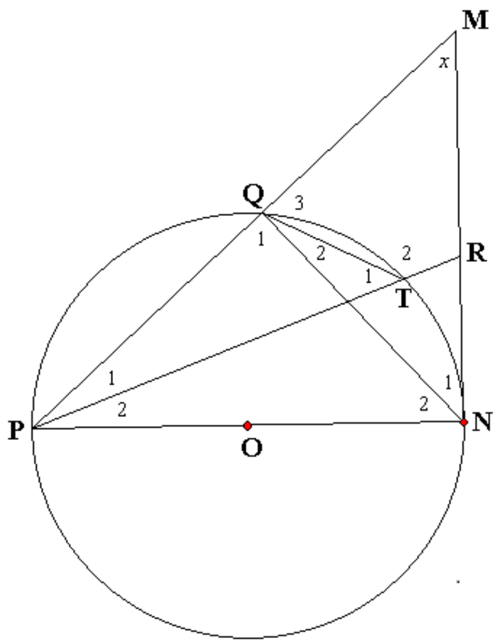


8 In the diagram above, PN is the diameter of the circle. NRM is a tangent to the circle at N. P, N, T and Q lie on the circle and $\hat{M} = x$. O is the centre of the circle.

8.1 Name, without reasons, THREE angles each equal to 90°

8.2 Is ΔPQN ||| ΔPNM? Justify your answer.

8.3 Name, with reasons, TWO other angles equal to *x*.

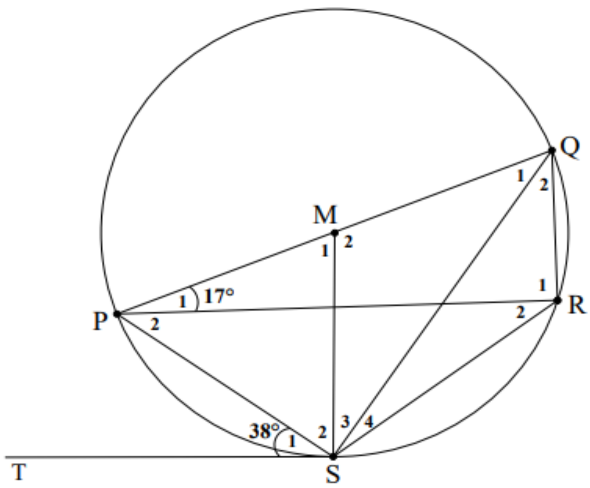


QUESTION 7

7.1 Complete the following theorem statement:

The angle between the tangent to a circle and the chord drawn from the point of contact is equal to … (1)

7.2 In the diagram below, PQ is the diameter of circle PQRS with centre M.
TS is the tangent to the circle at point S.
$\hat{S}_1 = 38°$ and $\hat{P}_1 = 17°$



Determine, with reasons, the sizes of:

7.2.1 $\hat{R}_2$ (2)

7.2.2 $\hat{M}_1$ (2)

7.2.3 $\hat{S}_2$ (2)

7.2.4 $\hat{Q}_2$ (5)

7.2.5 Give a reason why PM is not parallel to SR. (1)

[13]

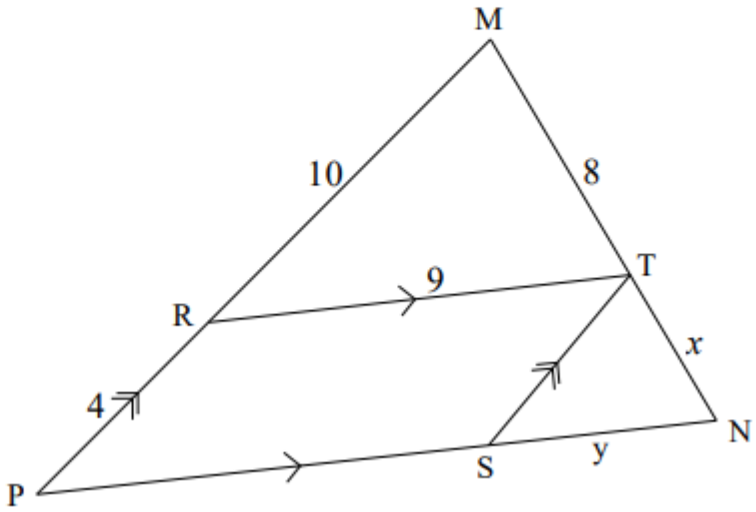
QUESTION 8

8.1 Complete the following theorem statement:

A line drawn parallel to one side of a triangle … (1)

8.2 In the diagram ΔMNP with R on MP and T on MN is given such that $RT \parallel PN$. S is a point on PN such that $TS \parallel MP$.

MR = 10 units
RP = 4 units
MT = 8 units
RT = 9 units
TN = x units
SN = y units



8.2.1 Calculate, stating reasons, the numerical value of x. (3)

8.2.2 What type of quadrilateral is RTSP? Give a reason for the answer. (2)

8.2.3 Hence calculate, stating reasons, the numerical value of y. (3)

8.2.4 Hence, show with calculations, that $\Delta MRT \parallel\parallel \Delta TSN$. (4)

[13]

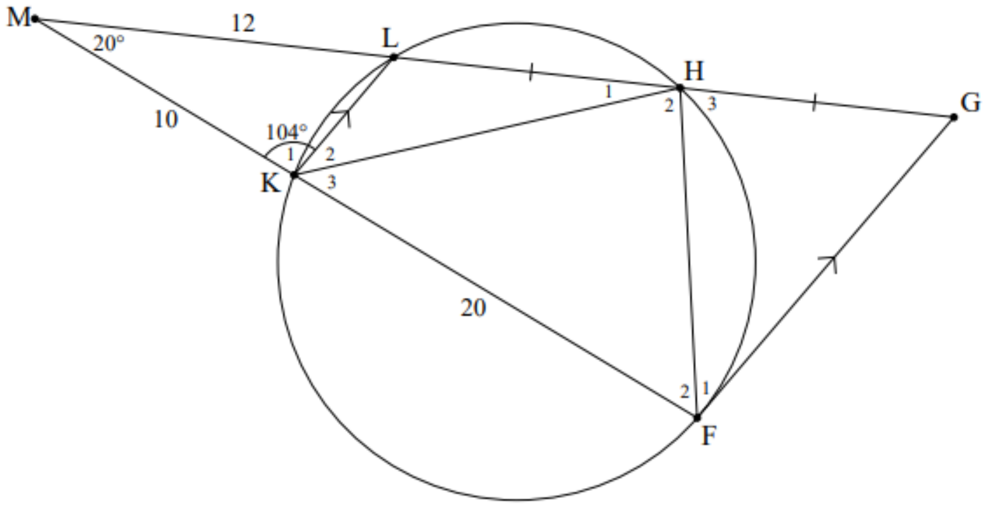
QUESTION 9

In the diagram, HLKF is a cyclic quadrilateral. The chords HL and FK are produced to meet at M. The line through F, parallel to KL, meets MH produced at G.

MK = 10 units
KF = 20 units
ML = 12 units
LH = HG

$\hat{M} = 20^\circ$

$\hat{K}_1 = 104^\circ$



9.2.1 Determine, stating reasons, the lengths of the following:

(a) QM (2)

(b) KP (3)

(c) KB (3)

9.2.2 (a) Give TWO reasons why $\Delta KBQ \parallel\!\mid \Delta KPM$. (2)

(b) Hence, or otherwise, determine the length of BQ if $PM = \sqrt{141}$ units. Leave your answer in simplified surd form. (3)

[14]

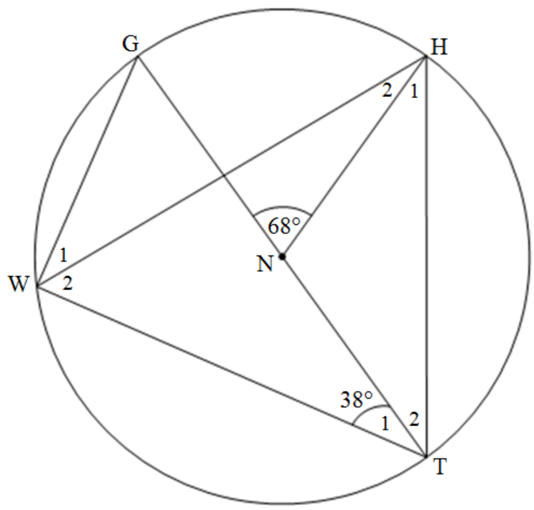
MAY / JUNE 2019

QUESTION 7

7.1 Complete the following theorem:
The angle subtended by the diameter at the circumference of the circle is … (1)

7.2 The diagram below shows circle GWTH with centre N.
GT is a diameter of the circle.

$G\hat{N}H = 68°$ and $\hat{T}_1 = 38°$



7.2.1 Determine, stating reason(s), the size of $\hat{W}_1$. (2)

7.2.2 Give a reason why $\hat{H}_1 = \hat{T}_2$ (1)

7.2.3 Hence, determine (stating reasons) the size of $\hat{H}_2$. (4)

[8]

QUESTION 8

8.1 Complete the following theorems:

8.1.1 The exterior angle of a cyclic quadrilateral is equal to … (1)

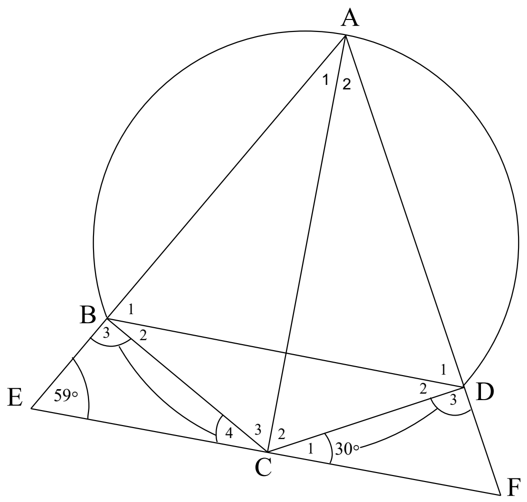
8.1.2 The angle between the tangent to a circle and the chord drawn from the point of contact is … (1)

8.2 The diagram below shows circle ABCD with AB produced to E and AD produced
to F.

ECF is a tangent to the circle at C and CA bisects $E\hat{A}F$.

$\hat{C}_1 = 30°$

$\hat{E} = 59°$



8.2.1 Give, with reasons, THREE other angles, each equal to $30°$. (5)

8.2.2 Determine, with reason(s), the size of $\hat{B}_1$ if $\hat{D}_1 = 61°$ (2)

8.2.3 Give a reason why BD || EF. (1)

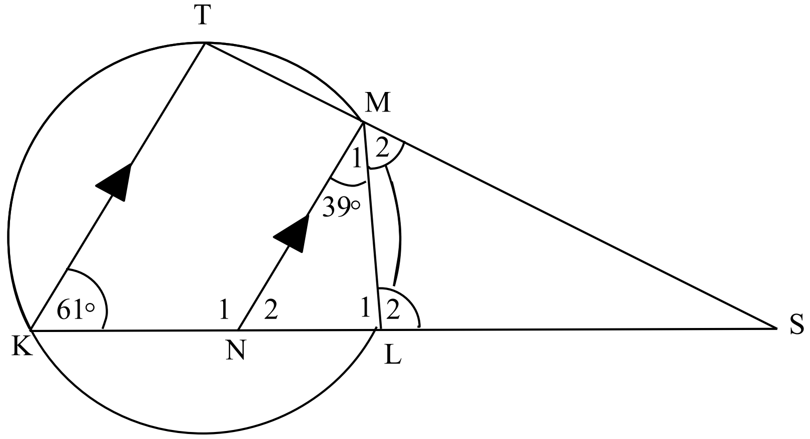
8.2.4 Determine, with reason(s), whether AC is a diameter of circle ABCD. (3)

8.3 The diagram below shows circle TKLM with chords TM and KL produced to meet at S.

TK || MN with N, a point on KL.

$\hat{K} = 61°$

$\hat{M}_1 = 39°$



8.3.1 Calculate, with reasons, the sizes of the following angles:

(a) $\hat{T}$ (3)

(b) $\hat{L}_1$ (2)

8.3.2 Show, with reasons, whether MS is a tangent to circle MNL. (3)

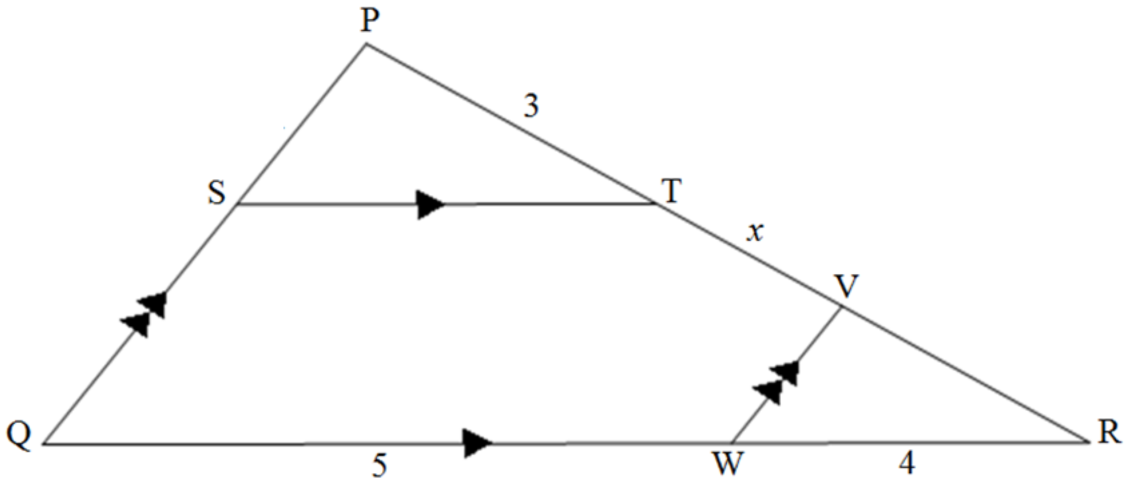
[21]

QUESTION 9

9.1 Complete the following theorem:

If a line divides two sides of a triangle in the same proportion, then the line is … (1)

9.2 In the diagram below, Δ PQR is drawn with S on PQ, T and V on PR and W on QR.
ST || QR and VW || PQ.
Furthermore, PS : SQ = 1 : 3
RW = 4 units, QW = 5 units, PT = 3 units and TV = x units.



9.2.1 Determine, with reason(s), the length of TR. (3)

9.2.2 (a) Express VR in terms of x. (1)

(b) Give the numerical value of $\frac{RV}{VP}$. (1)

(c) Hence, determine the numerical value of x. (3)

[9]

STUDY TIPS

A learner needs to **learn the diagram of the theorem first**. In my opinion, this is the most important step! This is the step that is very often brushed over very quickly, but it is the step that develops the “*Geometric eye*“. It is the step that will help a learner SEE the geometry since they know visually what they are looking for. If the learner does not know what it could look like when the theorem is applicable, how on Earth are they going to be able to see when to use it?!!

This is where the learner will learn about properties, i.e. **learning the statement of the theorem and the reason to be written next**. Since the learner has already learnt the diagram, the statement (and reason) that they will use makes more sense since the statement refers to what happens in the diagram! This will mean that the linking in the brain of the learner will be easier and therefore, remembering it will be easier.
There are many ways that can be used to remember the statements of the theorems. Below, there is an example of using a song to remember.

This is where another crucial difference in this method appears! The examples start here. The learner will **start with the simple numerical examples**. The purpose of this is that numerical examples can be done via simple deduction or informal deduction. They are generally not multi-step calculations (and if they are, they are still short). This helps the learner to be able to practise their “*Geometric eye*” by identifying the necessary theorem using the diagram they have learnt. Then they can further practise the application of that theorem.

MAY/JUNE 2021

QUESTION 6

The diagram below shows farmland in the form of a cyclic quadrilateral , PQRS.

The land has the following dimensions:

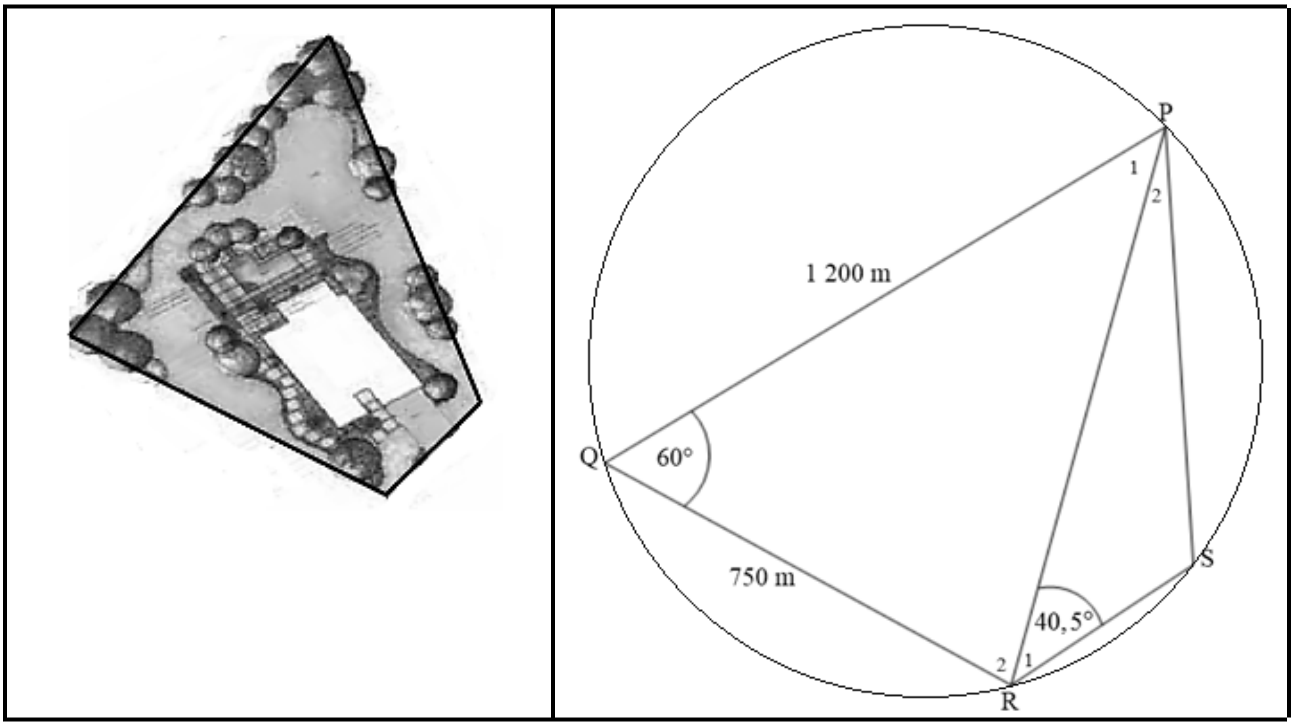
$PQ = 1\,200$ m

$QR = 750$ m

$\hat{Q} = 60°$

$\hat{R}_1 = 40,5°$

P, Q , R and S lie on the same horizontal plane.



Determine:

6.1 The length of PR (3)

6.2 The size of $\hat{S}$ (1)

6.3 The length of PS (3)

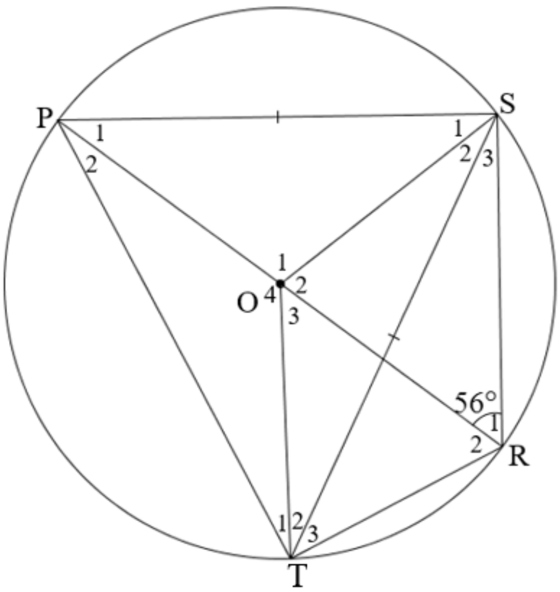
6.4 The area of ΔQPR (3)
[10]

QUESTION 7

7.1 Complete the following theorem statement:

Angles subtended by a chord of the circle, on the same side of the chord … (1)

7.2 In the diagram below, circle PTRS, with centre O, is given such that PS = TS.
POR is a diameter, OT and OS are radii.
$\hat{R}_1 = 56°$



7.2.1 Determine, stating reasons:

(a) Three other angles each equal to $56°$ (5)

(b) The size of $\hat{P}_1$ (3)

(c) The size of $\hat{S}_3$ (3)

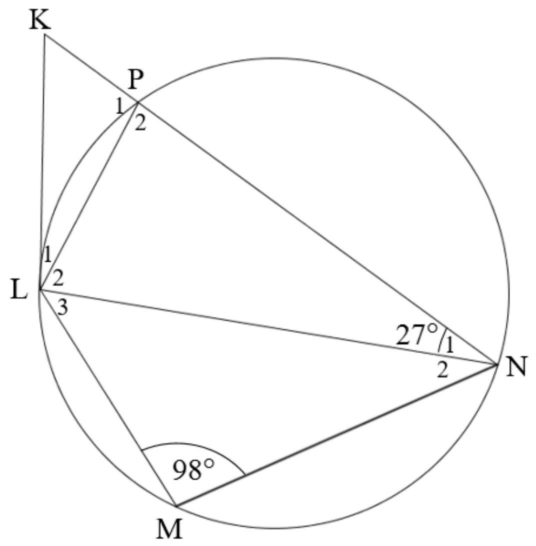
7.2.2 Prove, stating reasons, that OT is NOT parallel to SR. (3)

[15]

QUESTION 8

The diagram below shows circle LMNP with KL a tangent to the circle at L.
LN and NPK are straight lines.

$\hat{N}_1 = 27°$ and $\hat{M} = 98°$



8.1 Determine, giving reasons, whether line LN is a diameter. (2)

8.2 Determine, stating reasons, the size of:

8.2.1 $\hat{P}_2$ (2)

8.2.2 $\hat{P}_1$ (2)

8.2.3 $\hat{L}_1$ (2)

8.3 Prove, stating reasons, that:

8.3.1 Δ KLP ||| Δ KNL (3)

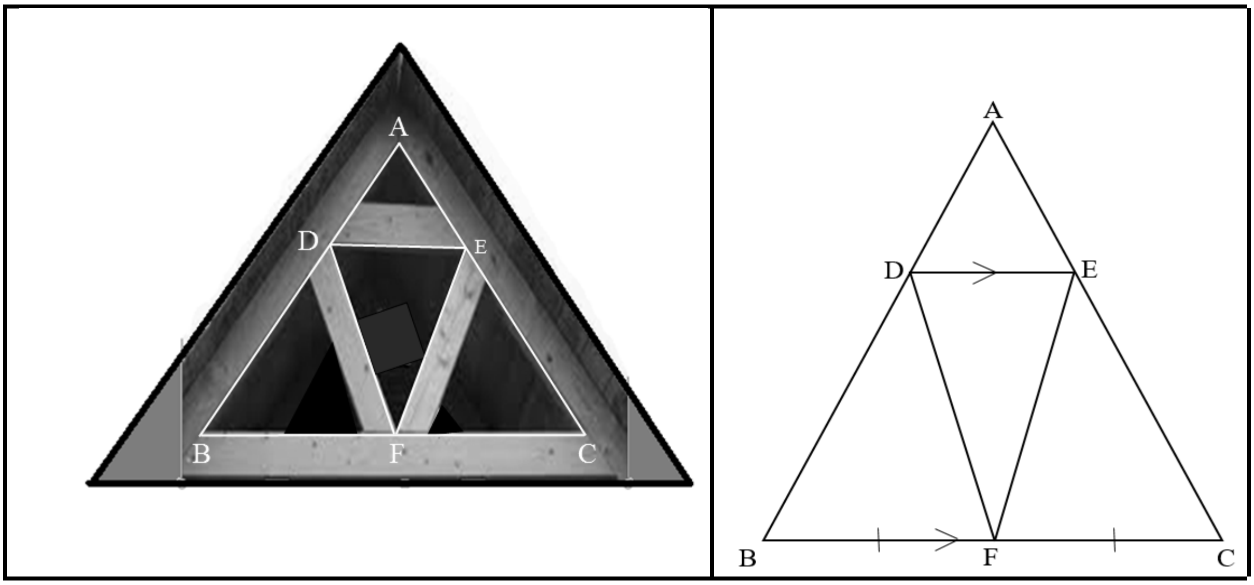
8.3.2 $KL^2 = KN \cdot KP$ (2)

8.4 Hence, determine the length of KP if it is further given that KL = 6 units and KN = 13 units. (2)

8.5 Determine, giving reasons, whether KLMN is a cyclic quadrilateral. (3)
[18]

QUESTION 9

The diagram below is a picture of a triangular roof truss, as shown.
Triangle ABC has AB = AC.
DE || BC and F is the midpoint of BC.
AE : EC = 1 : 2 and AB = 1,8 m.



9.1 Determine the length of:

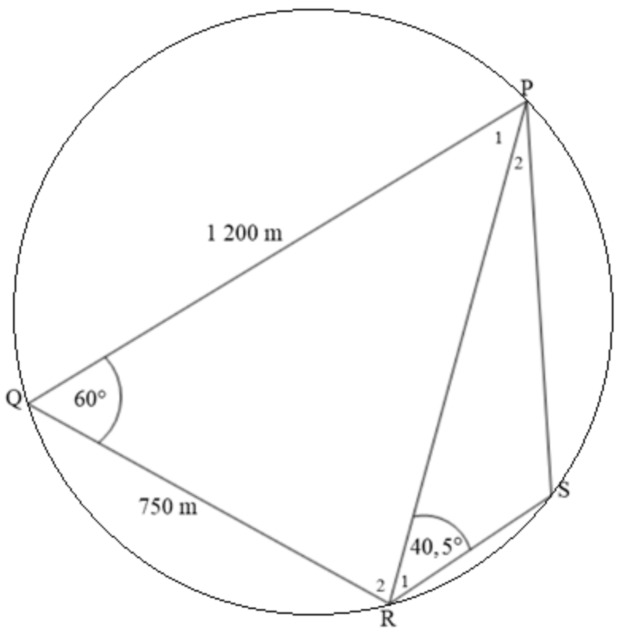
9.1.1 DB, giving reason(s) (2)

9.1.2 DF if DF = $\frac{3}{2}$AD (2)

9.2 Determine, giving reasons, whether EF is parallel to AB. (3)

[7]

QUESTION/*VRAAG* 6

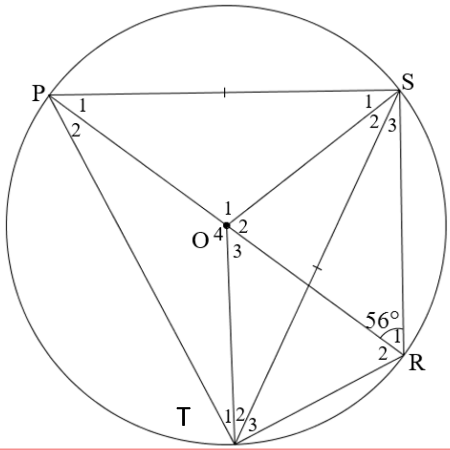
		
6.1	$PR^2 = QR^2 + PQ^2 - 2QR \cdot PQ \cos Q$ $= (750)^2 + (1200)^2 - 2(750)(1200)\cos 60°$ $= 1\ 102\ 500$ $\therefore PR = 1\ 050$ m	✓cosine rule/ *reël* **A** ✓**SF** **A** ✓value/PR/*wrde* **CA** (3)

6.2	$\hat{S} = 120°$	✓size of $\hat{S}$ **A** (1)
6.3	$\frac{PS}{\sin R_1} = \frac{PR}{\sin S}$ $\frac{PS}{\sin 40,5°} = \frac{1\ 050}{\sin 120°}$ $PS = \frac{1\ 050 \sin 40,5°}{\sin 120°}$ $\therefore PS \approx 787,41$m	✓sine rule/ *reël* **A** ✓ SF ✓value of PS/ *waarde van* **CA** **NPR** (3)
6.4	$\text{Area of } \Delta QPR = \frac{1}{2} QR \cdot QP \sin Q$ $= \frac{1}{2}(750)(1\ 200)\sin 60°$ $\approx 389711,43 \text{ m}^2$	✓ area rule/*reël* A ✓ SF A ✓value of/ *waarde* (3) **[10]**

QUESTION/*VRAAG* 7

7.1	are equal/ *is gelyk*	✓answer/ *antwoord* **A** (1)

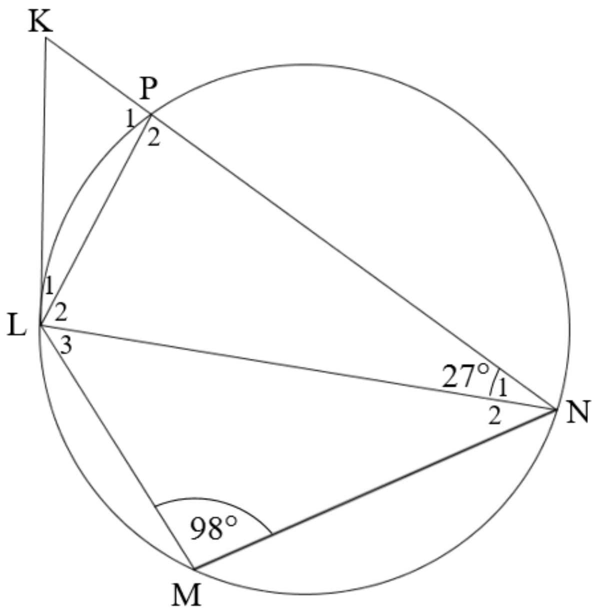
7.2



7.2.1(a)	$P\hat{T}S = \hat{R}_1 = 56°$ ($\angle$s in the same segment/ *dieselfde segment*) $O\hat{S}R = \hat{R}_1 = 56°$ ($\angle$s opp. = sides) /($\angle^e$ *teenoorg = sye*) $T\hat{P}S = P\hat{T}S = 56°$ [$\angle$s opp. = sides/*teenoorg = sye* **OR**/OF = chords subtend/ = *koorde onderspan* = $\angle$s]	✓ST **A** ✓RE **A** ✓ST **A** ✓RE **A** ✓ST **A** (5)
7.2.1(b)	$P\hat{S}R = 90°$ ($\angle$s in semicircle) /($\angle^e$ *in halfsirkel*) $\hat{P}_1 + 90° + 56° = 180°$ (sum of $\angle$s of Δ) /(*som van* $\angle^e$ *in* Δ) $\therefore \hat{P}_1 = 34°$	✓ST **A** ✓RE **A** ✓value of / *waarde van* $\hat{P}_1$ **CA** (3)

7.2.1(c)	$34° + \hat{P}_2 = 56°$ $\therefore \hat{P}_2 = 22°$ $\hat{S}_3 = \hat{P}_2 = 22°$ ($\angle$s in same segment) /($\angle^e$ *in dieslfde segment*) **OR/*OF*** $\hat{S}_1 + \hat{S}_2 + \hat{S}_3 = 90°$ ($\angle$ in the semicircle) /($\angle^e$ *in halfsirkel*) $\hat{S}_1 + \hat{S}_2 = 180° - 112°$ (sum of $\angle$s of Δ) /(*som van* $\angle^e$ *in* Δ) $= 68°$ $\therefore \hat{S}_3 = 90° - 68° = 22°$	✓ST **CA** ✓ST **CA** ✓RE **A** **OR/*OF*** ✓ST/RE **CA** ✓ST/RE **A** ✓ST **CA** (3)
7.2.2	$\hat{O}_3 = 44°$ ($\angle$ at centre $= 2 \times \angle$ at circum.) /(*mdpts* $\angle = 2 \times$ *omtrk* $\angle$) $\hat{O}_3 \neq \hat{R}$ $\therefore$ OT is not parallel to SR (alt. $\angle$s are not equal) $\therefore$ *OT is nie parallel an SR (verw* $\angle^e$ *nie gelyk)*	✓ST **A** ✓RE **A** ✓RE **A** (3)
		[15]

QUESTION/*VRAAG* 8

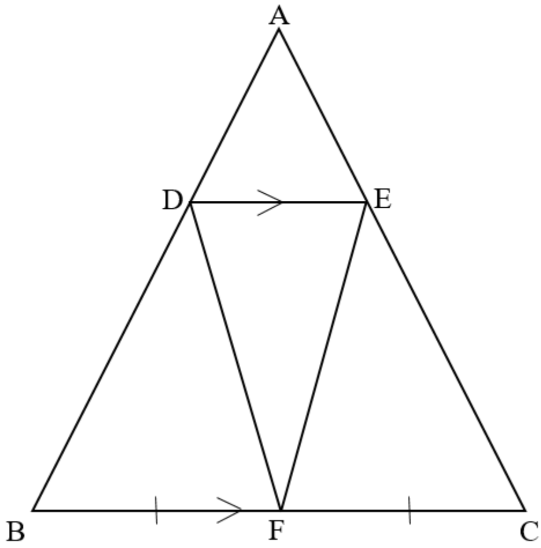


8.1	$\hat{M} = 98° \neq 90°$ $\therefore$ LN is not a diameter ($\angle$ subtended by LN $\neq 90°$) $\therefore$ *LN is nie'n middellyn* ($\angle$ *deur LN onderspan* $\neq 90°$) **OR/*OF*** $\hat{P}_2 + 98° = 180°$ (Opp. $\angle$s of cyclic quad.) /(*teens* $\angle^e$ *KVHK*) $\hat{P}_2 = 82° \neq 90°$ $\therefore$ LN is not a diameter ($\angle$ subtended by LN $\neq 90°$) $\therefore$ *LN is nie'n middellyn* ($\angle$ *deur LN onderspan* $\neq 90°$)	✓ST $\hat{M} = 98° \neq 90°$ **A** ✓RE **A** **OR/*OF*** ✓ST $\hat{M} = 98° \neq 90°$ **A** ✓RE **A** (2)
8.2.1	$\hat{P}_2 + 98° = 180°$ (Opp. $\angle$s of cyclic quad.) (*teenst* $\angle^e$ *van'n KVHK*) $\hat{P}_2 = 82°$	✓ST **A** ✓RE **A** (2)

8.2.2	$\hat{P}_1 + 82° = 180°$ ($\angle$s on straight line) /($\angle op'n\ r.lyn$) $\therefore \hat{P}_1 = 98°$ **OR/*OF*** $\hat{P}_1 = 98°$ (Ext. $\angle$ of a cyclic quad.) /($buite\ \angle van\ kvhk$)	✓ST **A** ✓RE **A** **OR/*OF*** ✓ST **A** ✓RE **A** (2)
8.2.3	$\hat{L}_1 = 27°$ (tan-chord theorem) / $rkl.koord\ st.$	✓ST **A** ✓RE **A** (2)

8.3.1	$\hat{K}$ is common $\hat{L}_1 = \hat{N}$ (both = 27°) $\hat{P}_1 = K\hat{L}N$ (3rd $\angle$ of Δ) $\therefore \Delta KLP \lvert\lvert\lvert \Delta KNL$ $(\angle,\angle,\angle)$ **OR** Equiangular/*gelykhoekig*	✓ST/RE **A** ✓ST/RE **A** ✓ST/RE **A** (3)
8.3.2	$\frac{KL}{KN} = \frac{KP}{KL}$ ($\lvert\lvert\lvert \Delta$s) $\therefore KL^2 = KN.KP$	✓ST **A** ✓RE **A** (2)
8.4	$KL^2 = KN.KP$ $(6)^2 = 13.KP$ $\therefore KP \approx 2{,}77$ units	✓ST **A** ✓value of / *waarde van* KP **A** (2)
8.5	$\hat{K} + 27° + 98° = 180°$ ($\angle$s of Δ) $\therefore \hat{K} = 55°$ $\hat{K} + \hat{M} = 55° + 98° \neq 180°$ $\therefore$ KLMN is not a cyclic quad. (Opp. $\angle$s are not suppl.) $\therefore$ *KLMN is nie 'n kvhk nie* (*teenst* $\angle^e$ *nie* $= 180°$) **OR/OF** L, M and N are on the circumference of the circle therefore KLMN is not a cyclic quad. K is outside the circle. L, M en N lê op die omtrek van die sirkel dus is KLMN nie 'n koordevierhoek. K lê buite die sirkel.	✓ST/RE **A** ✓value of / *waarde van* K **A** ✓RE **A** **OR/OF** ✓✓✓ST (3)
		[18]

QUESTION/*VRAAG* 9



9.1.1	$\frac{1,8}{DB}=\frac{3}{2}$ (Prop. theorem; DE || BC) $\therefore DB = 1,2\,m$ **OR/*OF*** $DB = \frac{2}{3} \times AB$ $= \frac{2}{3} \times 1,8\,m$ $\therefore DB = 1,2\,m$	✓ST/RE **A** ✓length of / *lente van* DB **A** **OR/*OF*** ✓M **A** ✓length of/ *lente van* DB **A** (2)
9.1.2	$AD = \frac{1,8}{3} = 0,6\,m$ $\therefore$ DF = $\frac{3}{2}(0,6\,m) = 0,9\,m$	✓M **A** ✓length of/ *lente van* DF **A** (2)

9.2	$\frac{CF}{FB}=\frac{1}{1}=1$ (BF = FC; F is the midpoint of/ *is die middelpunt van* BC) $\frac{CE}{EA}=\frac{2}{1}=2$ $\therefore \frac{CF}{FB}\neq\frac{CE}{EA}$ $\therefore$ EF is NOT parallel to AB (sides are not prop.) **OR/*OF*** BF = FC; F is the midpoint of/ *mdpt van* BC $AE\neq EC$; ; E is NOT the midpoint of/ *is NIE die middelpunt van* AC $\therefore$ EF is NOT parallel to AB (FE not joining midpoints of two sides of a triangle/ *verbind nie twee middelpunte van 'n driehoek*)	✓ST **A** ✓ ST **A** ✓Conclusion/ *gevolg.* **CA** **OR/*OF*** ✓ F is the midpoint of/ *mdpt van* BC **A** ✓ E is NOT the midpoint of/ *is NIE die middelpunt van* AC **A** ✓Conclusion/ *gevolg* **CA** (3)
		[7]

3. EXAMINATION TIPS

- Always have relevant tools (Calculator, Mathematical Set, etc.)
- Read the instructions carefully.
- Thoroughly go through the question paper, check questions that you see you are going to collect a lot of marks, start with those questions in that order because you are allowed to start with any question but finish that question.
- Write neatly and legibly.

4. ACKNOWLEDGEMENT

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DBE Subject Specialist: Mlungiseleli Njomeni

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