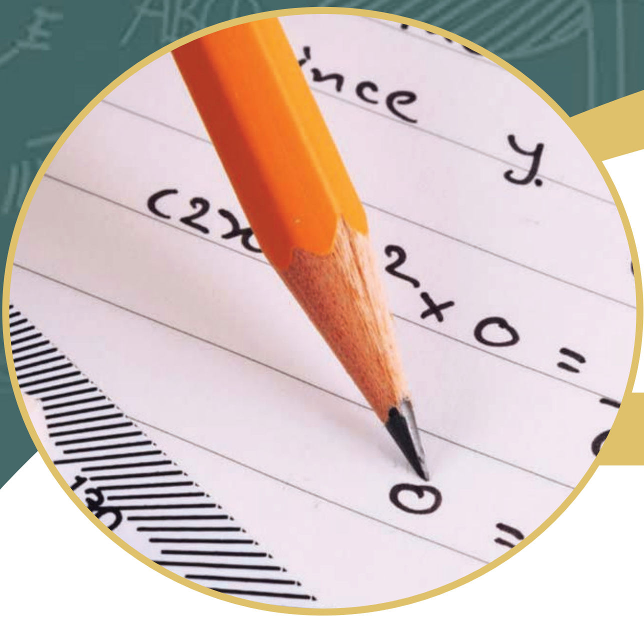


TECHNICAL MATHEMATICS

ANALYTICAL GEOMETRY

GRADE 12

basic education

Department:
Basic Education
REPUBLIC OF SOUTH AFRICA

Read to Lead
A Reading Nation is a Leading Nation

2030 NDP

Foreword

In order to improve learning outcomes the Department of Basic Education conducted research to determine the specific areas that learners struggle with in Grade 12 examinations. The research included a trend analysis by subject experts of learner performance over a period of five years as well as learner examination scripts in order to diagnose deficiencies or misconceptions in particular content areas. In addition, expert teachers were interviewed to determine the best practicesto ensure mastery of thetopic by learners and improve outcomes in terms of quality and quantity.

The results of the research formed the foundation and guiding principles for the development of the booklets. In each identified subject, key content areas were identified for the development of material that will significantly improve learner's conceptual understanding whilst leading to improved performance in the subject.

The booklets are developed as part of a series of booklets, with each bookletfocussing onlyon one specific challenging topic. The selected content is explained in detail and include relevant concepts from Grades 10 - 12 to ensure conceptual understanding.

The main purpose of these booklets is to assist learners to master the content starting from a basic conceptual level of understanding to the more advanced level. The content in each booklets is presented in an easy to understand manner including the use of mind maps, summaries and exercises to support understanding and conceptual progression. These booklets should ideally be used as part of a focussed revision or enrichment program by learners after the topics have been taught in class. The booklets encourage learners to take ownership of their own learning and focus on developing and mastery critical content and skills such as reading and higher order thinking skills.

Teachers are also encouraged to infuse the content into existing lesson preparation to ensure in-depth curriculum coverage of a particular topic. Due to the nature of the booklets covering only one topic, teachers are encouraged to ensure learners access to the booklets in either print or digital form if a particular topic is taught.

Table of Contents

2. How to use this booklet

This booklet is designed to clarify the content prescribed for Mathematics. In addition, it has some tips on how you should tackle real-life problems on a daily basis.

Candidates will be expected to have already mastered the content outlined for Grades 8-11. This booklet must be used to master some mathematical rules that you may not yet be aware of. The prescribed textbook must also be used.

3. Study and examination tips

All learners should be able to acquire sufficient understanding and knowledge to:

- develop fluency in computation skills without relying on the use of a calculator;
- generalise, make conjectures and try to justify or prove them;
- develop problem-solving and cognitive skills;
- make use of the language of Mathematics;
- identify, investigate and solve problems creatively and critically;
- use properties of shapes and objects to identify, investigate and solve problems creatively and critically;
- encourage appropriate communication by using descriptions in words, graphs, symbols, tables and diagrams; practise Mathematics every day.

4. Analytical Geometry

1. Distance formula

The distance formula is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ It is used to:

- Calculate the distance/ length, given two points.
- Determine the x-coordinate or y-coordinate, given the length.
- Show/prove that a triangle is:
  Isosceles: two sides are equal.
  Equilateral: all sides are equal.
  A right-angled triangle: the square of the hypotenuse equals the sum of the squares on the other two sides.
  Scalene: all sides are not equal.
- Show/ prove that a quadrilateral shape is a:
  Rectangle: opposite sides are equal.
  Parallelogram: opposite side are equal.
  Square: all sides are equal.
  Rhombus: all sides are equal.
  Kite: adjacent sides are equal.
  Scalene: all sides are not equal.

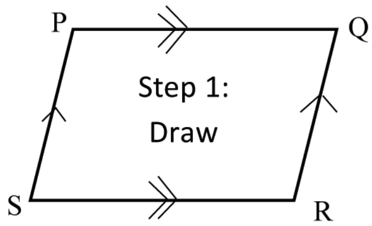
2. Midpoint of a line segment

The midpoint formula is $M\left(\frac{x_1 + x_2}{2}; \frac{(y_1 + y_2)}{2}\right)$ It is used to:

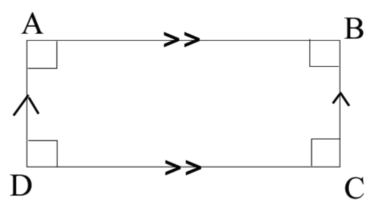
- Determine the midpoint of line segment.
- Determine the coordinates given the midpoint of a line segment.
- Determine the coordinates of the fourth vertex of:
- ✓ Rectangle
- ✓ Parallelogram
- ✓ Square
- ✓ Rhombus
- Show that the quadrilateral is:
- ✓ Rectangle
- ✓ Parallelogram
- ✓ Square
- ✓ Rhombus, since the mid-points of the diagonals bisect each other.

3. The gradient or slope is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$**.** It is used to:

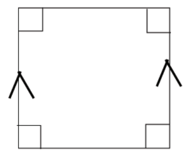
- Determine the gradient of the line segment.
- Show that two lines are parallel, since parallel lines have equal gradients, i.e. $m_1 = m_2$
- Show that two lines are perpendicular, since the product of their gradient is -1, i.e. $m_1 \times m_2 = -1$
- Show that points are collinear, i.e. points that lie on the same straight line have the same gradient/ slope.
- Show that a triangle is a right-angled triangle, since two sides are perpendicular.
- Show that a quadrilateral shape is a:
  - **Parallelogram:**
    - ✓ Two pairs of opposite sides are parallel; or
    - ✓ Diagonals bisect each other; or
    - ✓ Opposite angles are equal; or two pairs of opposite sides are equal; or
    - ✓ One pair of sides is parallel and equal.



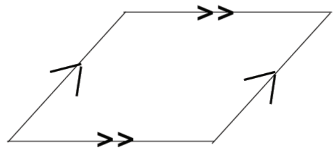
  - **Rectangle:**
    - ✓ Opposite sides are parallel and one interior angle is equal to 90^0.
    - ✓ Diagonals are equal in length and bisect reach other.



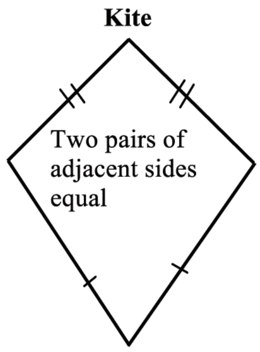
  - **Square:**
    - ✓ Parallelogram with at least one interior angle equal to 90º.
    - ✓ Diagonals equal in length and which bisect each other at 90°.



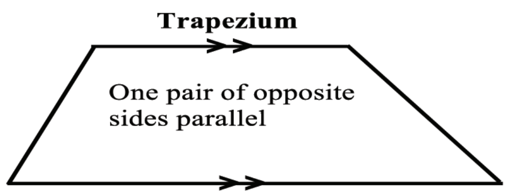
- **Rhombus:**
  - ✓ Parallelogram with one pair of adjacent sides equal.
  - ✓ Parallelogram in which the diagonals bisect at right angles.



- **Kite:**
  - ✓ Adjacent sides are equal in length.
  - ✓ The longer diagonal bisects the shorter diagonal at 90°.



- **Trapezium:**
  - ✓ One pair of opposite sides is parallel.

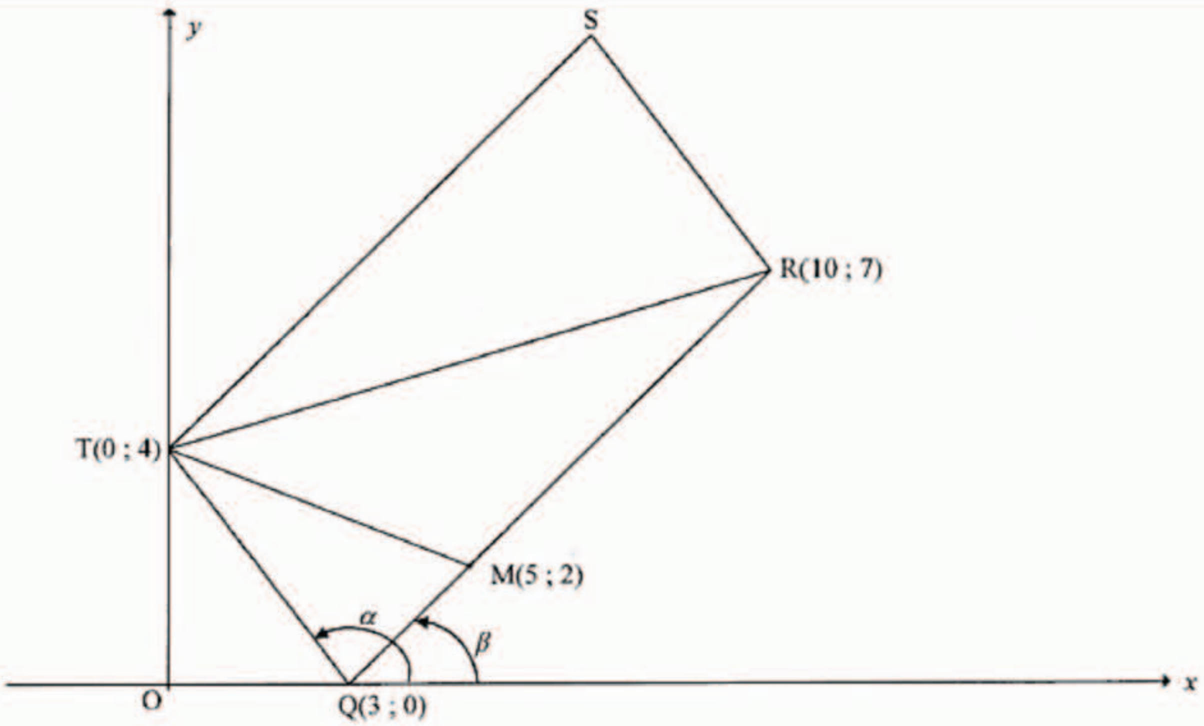


- Determine the inclination of a straight line: $\tan\theta = m$. Inclination is an angle formed between the straight line and the x-axis in the positive direction of the x-axis. It is always positive. The inclination of the lines can be used to find the size of the angles inside a triangle, using the following:
  - ✓ The sum of the angles of a triangle are supplementary (add up to 180°).

	✓ Vertically opposite angles are equal. ✓ The exterior angle of a triangle is equal to the sum of two opposite interior angles. • Find the equation of a straight line: $y - y_1 = m(x - x_1)$ • A line parallel to the x-axis has a zero gradient. • For a line parallel to the y-axis, the gradient is undefined. • Find the gradient of a straight line, given the inclination of that line.
4. The equation of a straight line: $y = mx + c$ or $y - y_1 = m(x - x_1)$	
	• Determine the equation of a straight line. • Determine the gradient of a straight line by expressing it in the form $y = mx + c$, where the coefficient of x is the gradient. • Show that the point lies on the straight line, where LHS = RHS. • Find the coordinates of the point of intersection of two lines, by solving them simultaneously. • Find the coordinates of the intercept on the axes. • Find the equation of the perpendicular bisector, using $m_1 \times m_2 = -1$. Find the equation of a tangent to the circle: $m_{rad} \times m_{\tan} = -1$ since the tangent is always perpendicular to the radius.

Example 1

In the diagram, $Q(3,0)$, $R(10;17)$, S and $T(0;4)$ are the vertices of the parallelogram QRST. From T, a straight line is drawn to meet QR at $M(5;2)$. The angles of inclination of TQ and RQ are α and β respectively.



1.1	Calculate the gradient of TQ.
1.2	Calculate the length of RQ. Leave your answer in surd form.
1.3	$F(k;-8)$ is a point on the Cartesian plane such that T, Q and F are collinear. Calculate the value of k.
1.4	Calculate the coordinates of S.
1.5	Calculate the size of $T\hat{S}R$.
1.6	Calculate, in the simplest form, the ratio of:
	1.6.1 $\dfrac{MQ}{RQ}$
	1.6.2 $\dfrac{\text{area of } \Delta TQM}{\text{area of parallelogram RQTS}}$

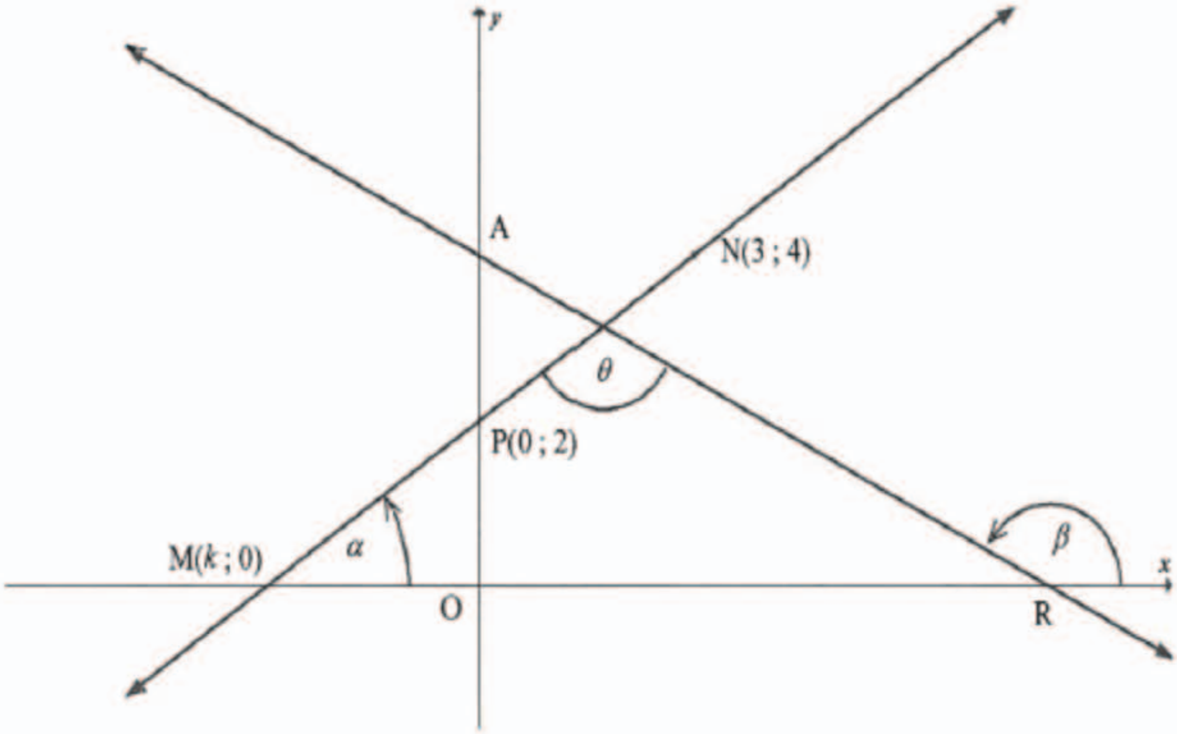
Solution:

1.1	$m_{TQ} = \dfrac{4-0}{0-3}$ $= -\dfrac{4}{3}$

1.2	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ $RQ = \sqrt{(10-3)^2 + (7-0)^2}$ $= \sqrt{98}$ $= 7\sqrt{2}$	
1.3	$m_{FQ} = m_{PQ}$ $\frac{-8}{k-3} = -\frac{4}{3}$ $4k - 12 = 24$ $k = 9$	
1.4	Midpoint of TR = midpoint of SQ [diag of $\|^{m}$] $\frac{x_s + 3}{2} = 5$ and $\frac{y_s + 0}{2} = \frac{11}{2}$ $\therefore x_s = 7$ and $y_s = 11$ $\therefore S(7;11)$	
1.5	$T\hat{S}R = T\hat{Q}R$ $T\hat{Q}R = \alpha - \beta$ $\tan\alpha = m_{TQ} = -\frac{4}{3}$ $\therefore \alpha = 180^0 - 53{,}13^0 = 126{,}87^0$ $\tan\beta = m_{PQ} = \frac{7}{7} = 1$ $\therefore \beta = 45^0$ $T\hat{Q}R = 126{,}87^0 - 45^0$ $= 81{,}87^0$ $T\hat{S}R = 81{,}87^0$	
	1.6.1	$MQ = \sqrt{(5-3)^2 + (2-0)^2}$ $MQ = \sqrt{8}$ $\frac{MQ}{RQ} = \frac{\sqrt{8}}{98}$ $\frac{2}{7}$
	1.6.2	$\frac{\text{area of } \Delta TQM}{\text{area of } \Delta TQR} = \frac{\frac{1}{2}.QM. \perp h}{\frac{1}{2}.QR \perp h}$ [same $\perp$ h] $= \frac{QM}{QR}$ $= \frac{2}{7}$

Example 2

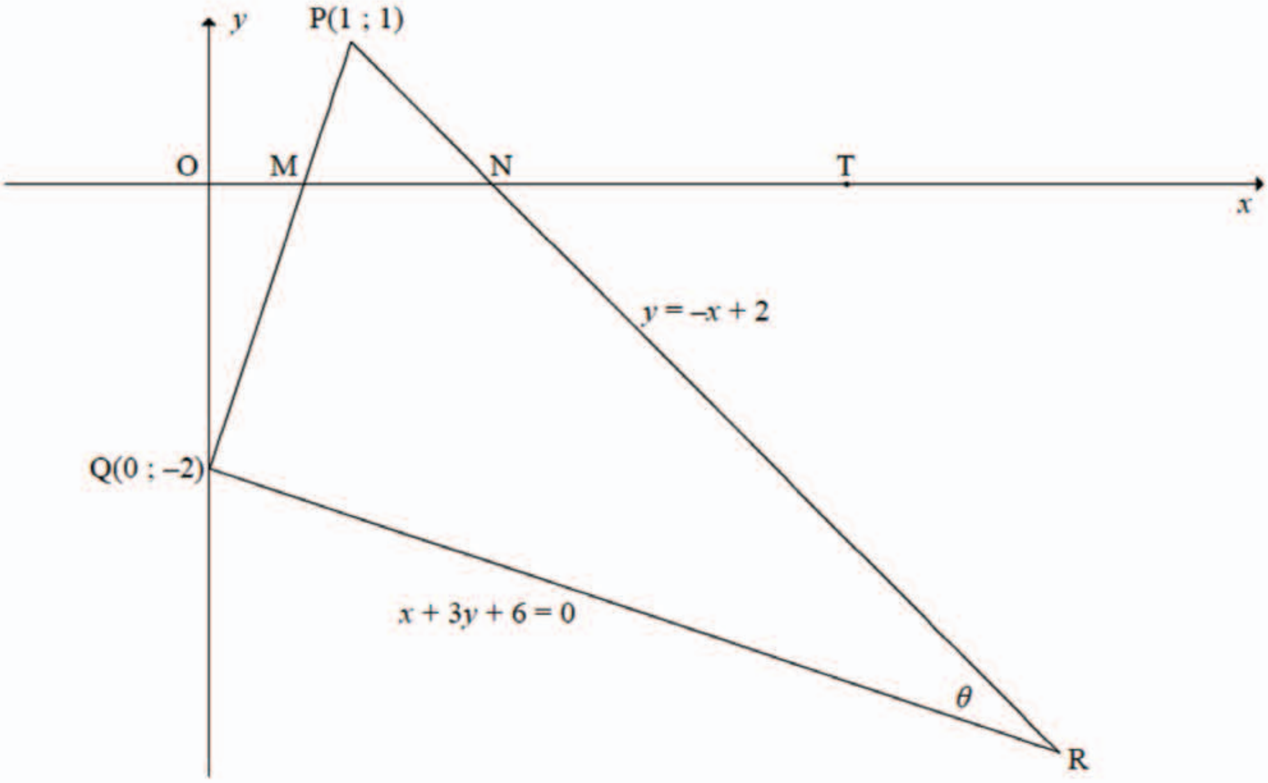
In the diagram, R and A are the x- ad y-intercepts respectively, of the straight line AR. The equation of AR is $y=-\frac{1}{2}x+4$. Another straight line cuts the y-axis at $P(0;2)$ and passes through the points $M(k;0)$ and $N(3;4)$. A and β are the angles of inclination of the lines MN and AR respectively. -

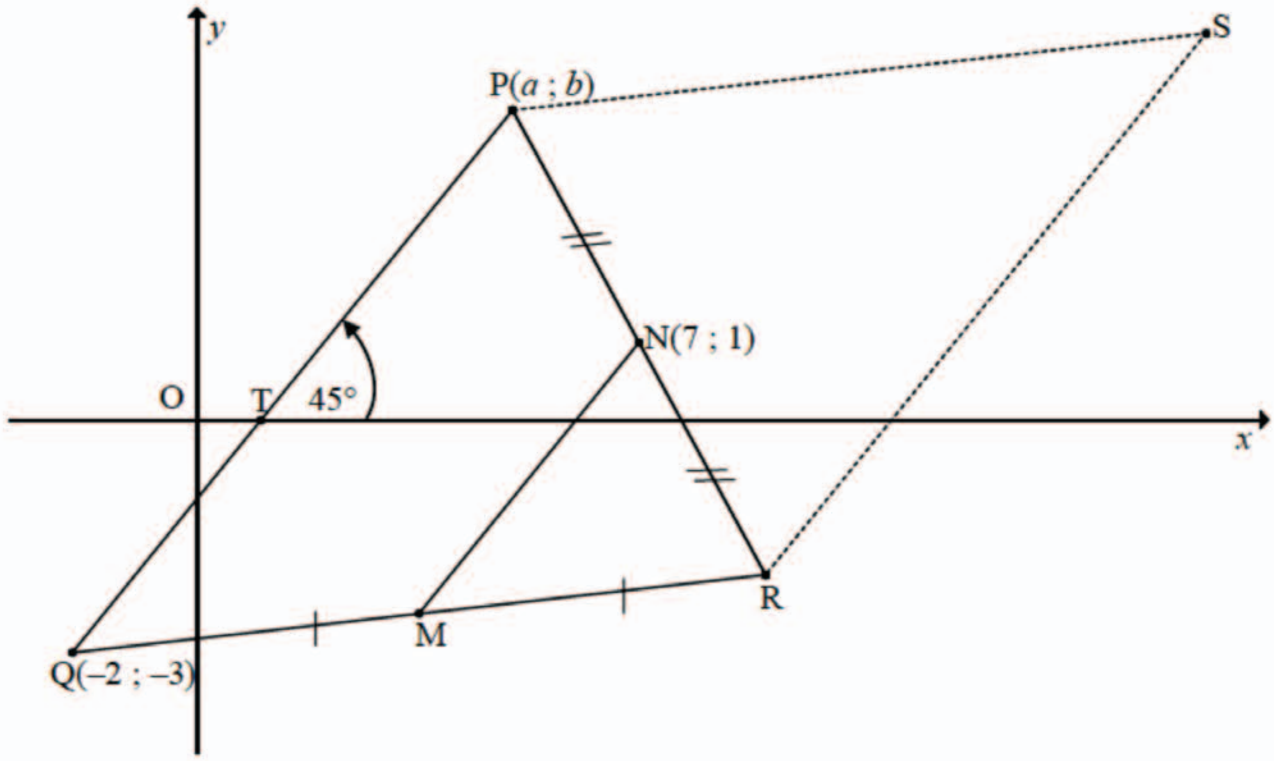


2.1	Given that M, P and N are collinear, calculate the value of k.
2.2	Determine the size of θ, which is the obtuse angle between the two lines.
2.3	Calculate the length of MR.
2.4	Calculate the area of Δ MNR.

Solution :	
2.1	$m_{MP} = m_{PN}$ $\frac{2-0}{0-k} = \frac{4-2}{3-0}$ $\frac{2}{-k} = \frac{2}{3}$ $k = -3$
2.2	$\tan\alpha = m_{PN}$ $\tan\alpha = \frac{2}{3}$ $\alpha = 33{,}69^0$ $\tan\beta = m_{AB}$ $\tan\beta = -\frac{1}{2}$ $\beta = -26{,}57^0 + 180^0$ $= 153{,}43^0$ $\theta = 153{,}43^0 - 33{,}69^0$ $= 119{,}74^0$
2.3	$-\frac{1}{2}x + 4 = 0$ $x = 8$ $R(8\,;0)$ $MR = 8-(-3)$ $= 11$ units
2.4	Area of $\Delta MNR = \frac{1}{2}(MR).\perp h$ $= \frac{1}{2}(11)(\text{y - value of N})$ $= \frac{1}{2(11)(4)}$ $= 22$ square units

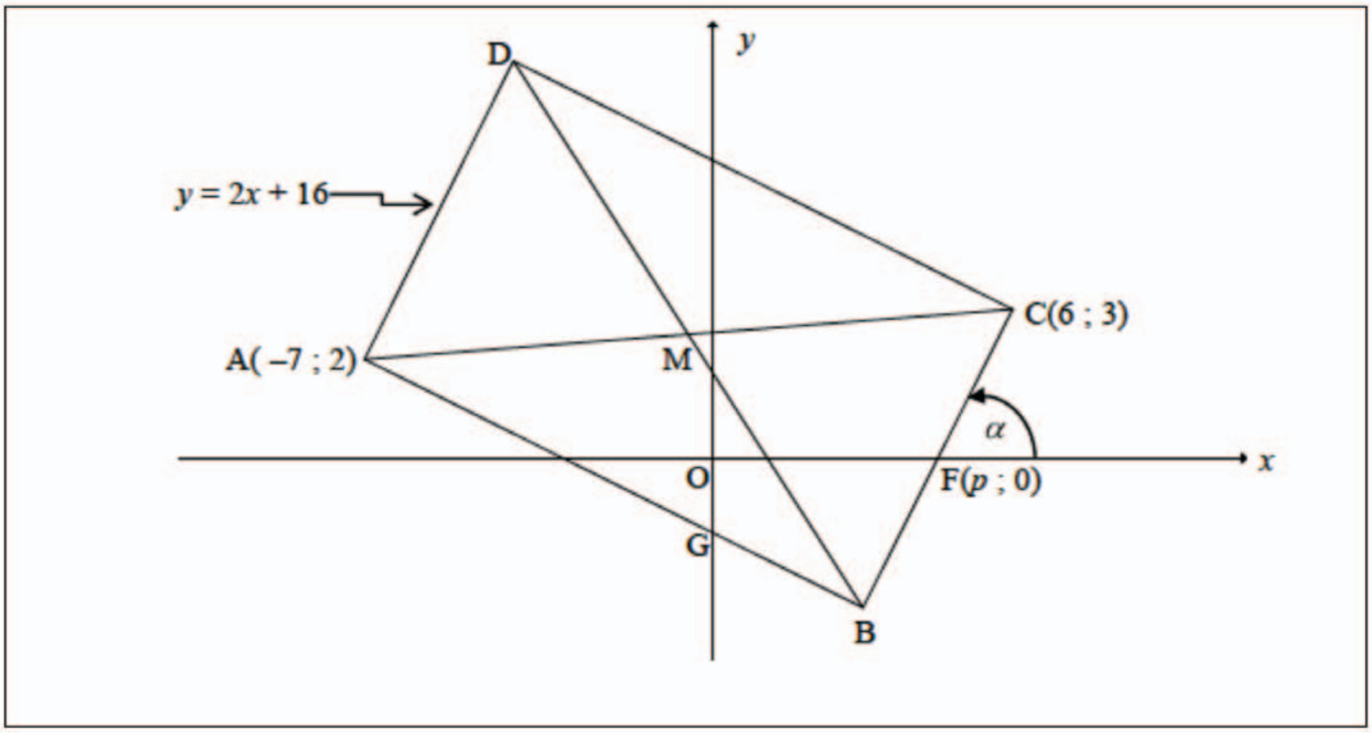
Activity 1

1.	In the diagram below, P(1 ; 1), Q(0 ; –2) and R are the vertices of a triangle and $P\hat{R}Q = \theta$. The x-intercepts of PQ and PR are M and N respectively. The equations of the sides PR and QR are $y = -x + 2$ and $x + 3y + 6 = 0$ respectively. T is a point on the x-axis, as shown below. 
1.1	Determine the gradient of QP.
1.2	Prove that $P\hat{Q}R = 90°$.
1.3	Determine the coordinates of R.
1.4	Calculate the length of PR. Leave your answer in surd form.
1.5	Calculate the size of θ.
2.	In the diagram below, the line joining Q(–2 ; –3) and P(a ; b), a and $b > 0$, make an angle of 45° with the positive x-axis. QP = $7\sqrt{2}$ units. N (7 ; 1) is the midpoint of PR and M is the midpoint of QR.



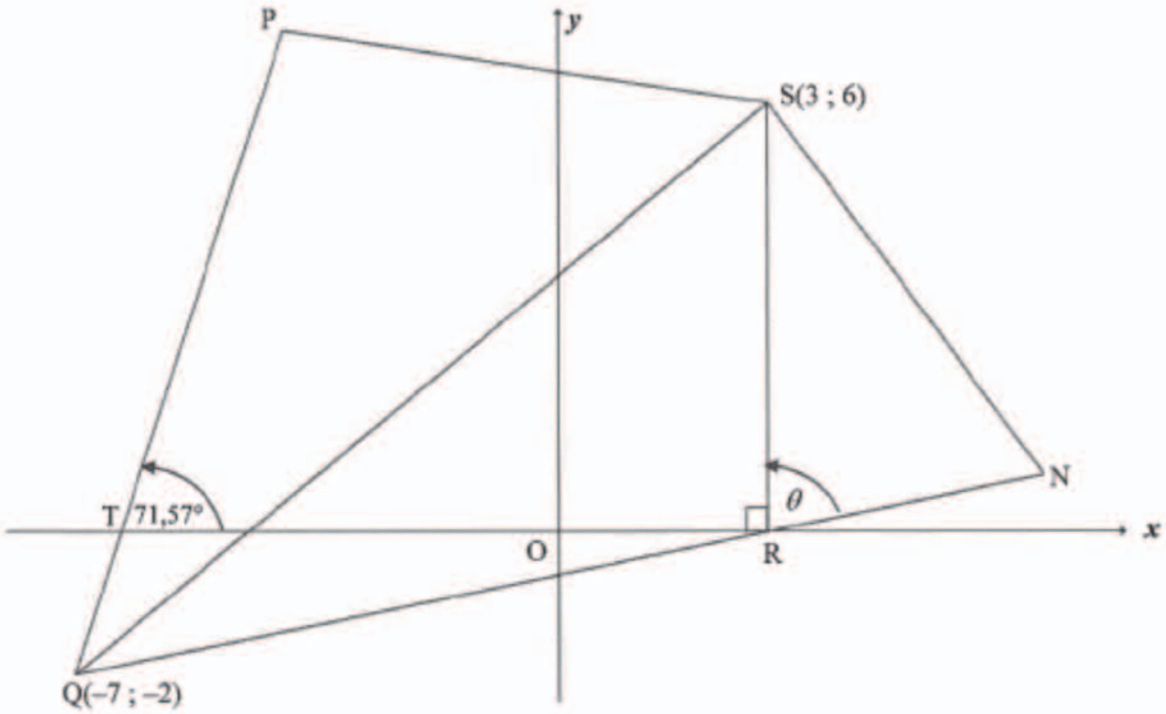
Determine:	
2.1	The gradient of PQ.
2.2	The equation of MN in the form y = $mx + c$ and give reasons.
2.3	The length of MN.
2.4	The coordinates of S such that PQRS, in this order, is a parallelogram.
2.5	The length of RS.
2.6	The coordinates of P.

3. In the diagram, A(–7 ; 2), B, C(6 ; 3) and D are the vertices of rectangle ABCD. The equation of AD is $y = 2x + 16$. Line AB cuts the y-axis at G. The x-intercept of line BC is F(p ; 0) and the angle of inclination of BC with the positive x-axis is α. The diagonals of the rectangle intersect at M.



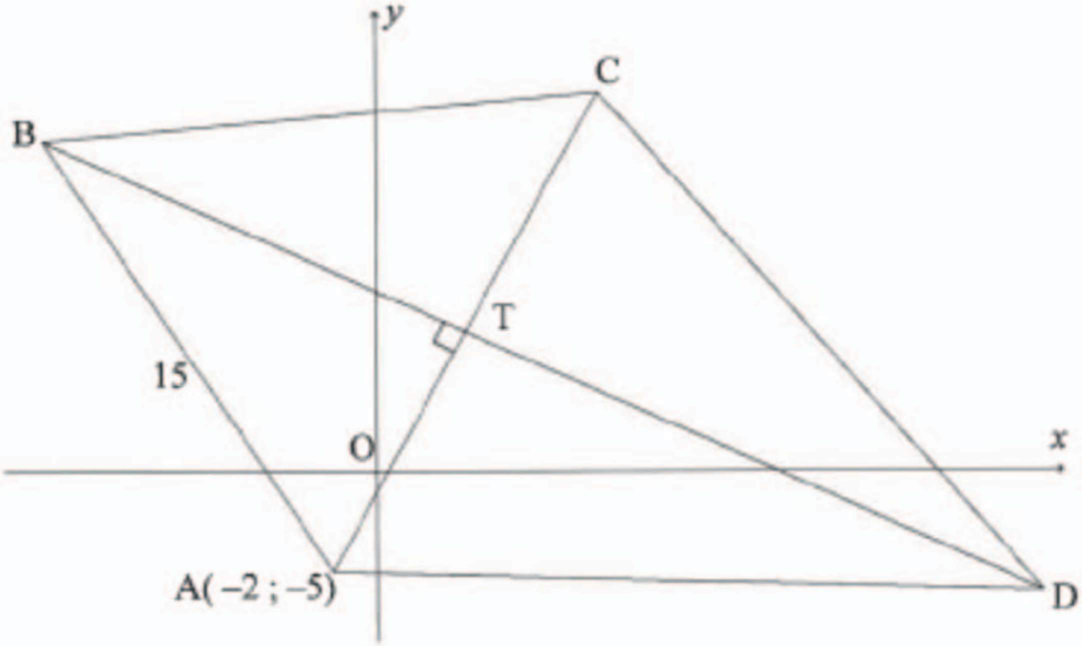
3.1	Calculate the coordinates of M.
3.2	Write down the gradient of BC in terms of p.
3.3	Hence, calculate the value of p.
3.4	Calculate the length of DB.
3.5	Calculate the size of α.
3.6	Calculate the size of $O\hat{G}B$.

4. In the diagram, P, $Q(-7;-2)$, R and $S(3;6)$ are vertices of a quadrilateral. R is a point on the x-axis. QR is produced to N such that QR = 2RN. SN is drawn. $P\hat{T}Q = 71{,}57^0$ and $S\hat{R}N = \theta$.



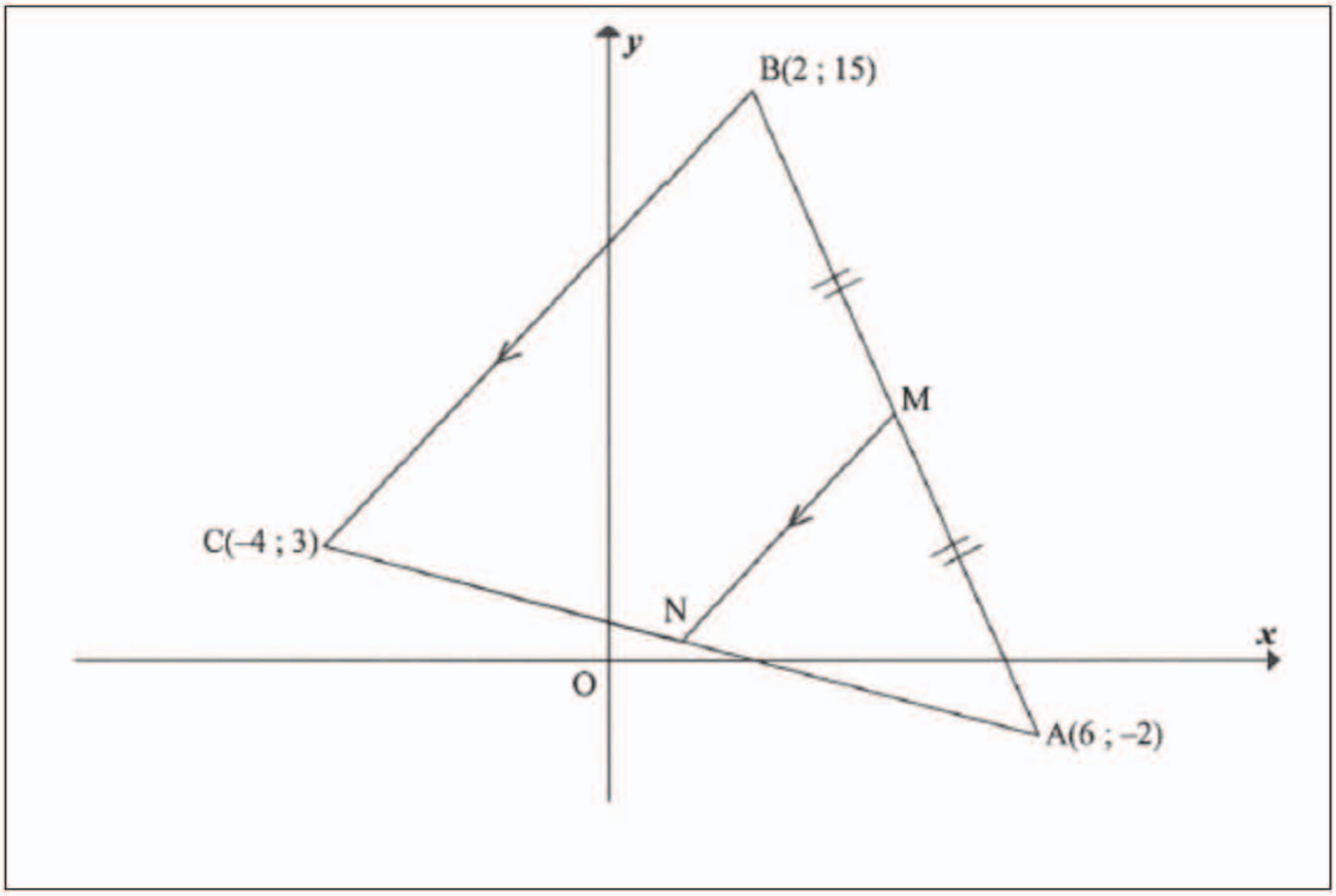
	Determine:
4.1	The equation of SR.
4.2	The gradient of QP to the nearest integer.
4.3	The equation of QP in the form $y = mx + c$
4.4	The length of QR. Leave your answer in surd form.
4.5	$\tan(90^0 - \theta)$
4.6	The area of ΔRSN, without using a calculator.

5. $A(-2;-5)$, B, C and D are the vertices of quadrilateral ABCD, such that diagonal AC is perpendicular to diagonal BD at T.
The equation of BTD is given by $2y + x = 18$ and AB=15 units.



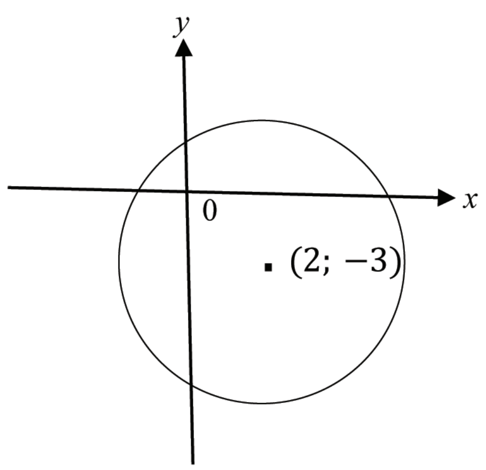
5.1	Determine the gradient of line AC.	
5.2	Determine the equation of AC in the form $y = mx + c$.	
5.3	If the equation of AC is $y = 2x - 1$, calculate the coordinates of T.	
5.4	ABCD is a kite with AB=BC.	
	5.4.1	Determine the coordinates of C.
	5.4.2	Calculate the length of BT.
	5.4.3	Write down the length of the radius of the circle passing through the points B, C and T.

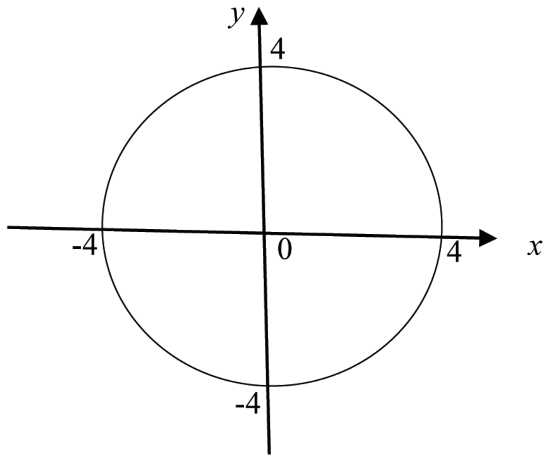
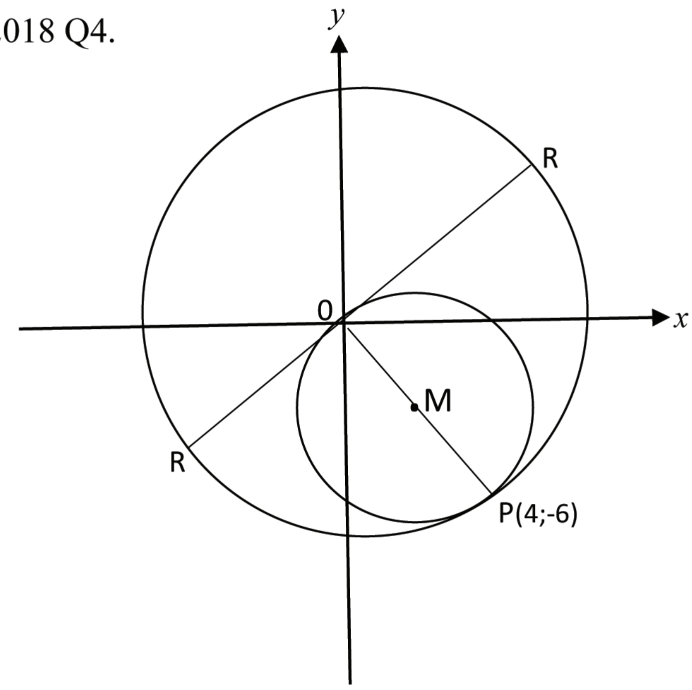
6. In the diagram, A (6;-2), B (2;15) and C (-4;3) are the vertices of ΔABC. M is the midpoint of AB. N is a point on CA such that MN| | BC.

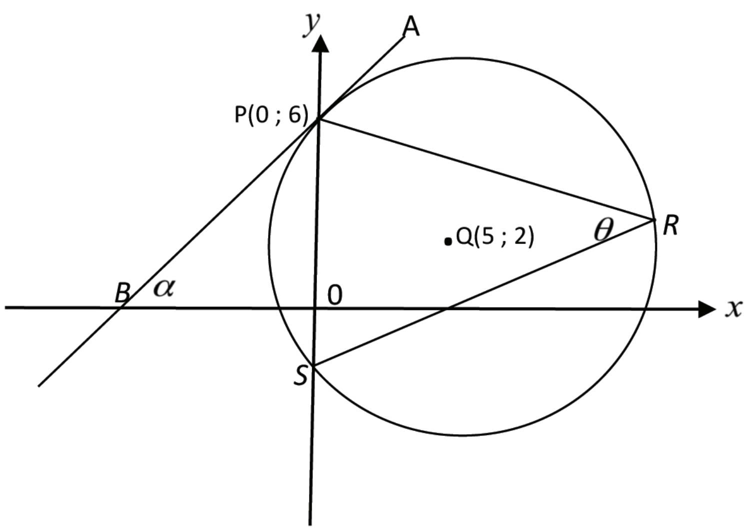


6.1	Determine the coordinates of M, which is the midpoint of AB.
6.2	Determine the gradient of line MN.
6.3	Hence, or otherwise, determine the equation of line MN, in the form $y = mx + x$.
6.4	Calculate, with reasons, the coordinates of point N.
6.5	If ABCD (in that order) is a parallelogram, determine the coordinates of point D.

	5. Equation of a circle • The equation of circle with the centre not at the origin. • The equation of a tangent to a circle.
5.1	**Concepts and skills**
	• Calculate the equation of a circle with centre $(a ; b)$ and radius r. • Determine the equation of a tangent to a given circle.
5.2	**Prior knowledge:**
	• Simplification • Writing equations in standard form

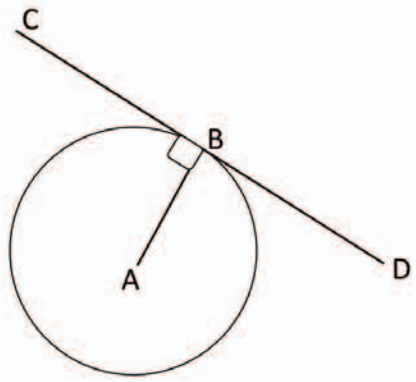
	• Completing squares in quadratic equations • Tangent • Parallel and perpendicular lines • Gradients • Factorisation
5.3	**Definitions:**
	A tangent is a line that touches a circle at one point. Perpendicular lines are lines that form a 90^0 angle between them. The product of their gradients is -1, i.e. $m_1 \times m_2 = -1$ Parallel lines are lines that are equidistant from each other. Their gradients are equal, i.e. $m_1 = m_2$
5.4	**Equation of a circle**
	• The general equation of a circle, where the centre is not at the origin, is given as: $(x-a)^2 + (y-b)^2 = (r)^2$ .... (1) where the centre of the circle is (a ; b). e.g. $(x-2)^2 + (y+3)^2 = 25$. The centre of the circle is $(2; -3)$ and the radius is 5.  • A circle with the centre at the origin is a special case. When substituting for (0 ; 0) in equation (1), the equation becomes $x^2 + y^2 = r^2$. • Consider the following: $x^2 + y^2 = 16$. This equation tells us that the centre of the circle is (0 ; 0) and the radius is 4, i.e. $(x-0)^2 + (y-0)^2 = (4)^2$

	
	Given the equations of a circle, write the equations from one form to the other and vice versa, e.g. SC 2018 Q4.  Given the coordinates of M a (2;-3), determine: • the equation of the large circle $x^2 + y^2 = (4)^2 + (-6)^2$ $= 52$ $\therefore x^2 + y^2 = 52$ • the equation of the small circle in the form $x^2 + y^2 + Cx + Dy + E = 0$ $(x-2)^2 + (y+3)^2 = \left(\frac{\sqrt{52}}{2}\right)^2 = 13$ $x^2 - 4x + 4 + y^2 + 6y + 9 = 13$ $x^2 + y^2 - 4x + 6y + 4 + 9 - 13 = 0$

	$x^2 + y^2 - 4x + 6y = 0$ **NB:** (If given in the form $x^2 + y^2 - 4x + 6y = 0$**:** we need to complete the square to arrive at $(x-2)^2 + (y+3)^2 = 13$) (If given in the form $(x-2)^2 + (y+3)^2 = 13$**:** we need to simplify to arrive at $x^2 + y^2 - 4x + 6y = 0$)
5.5.5	**Activity:**
	1. Given the equations of circles: 1.1 $x^2 + y^2 - 4x - 6y + 9 = 0$ 1.2 $x^2 + y^2 - 6x + 2y + 8 = 0$ • Rewrite the equation in the form $(x-a)^2 + (y-b)^2 = r^2$ • Give the coordinates of the centre of the circle. • Write down the radius of the circle.
	In the diagram below, Q (5 ; 2) is the centre of a circle that intersects the y-axis at P (0 ; 6) and S. The tangent APB at P intersects the x-axis at B and makes the angle α with the positive x-axis. R is a point on the circle and $P\hat{R}S = \theta$.
	

	Determine the equation of the circle in the form $(x-a)^2+(y-b)^2=r^2$.
	Calculate the coordinates of S.
	Determine the equation of the tangent APB in the form $y = mx + c$.
	Calculate the size of α.
	Calculate, with reasons, the size of θ.
	Calculate the area of ΔPQS.
5.5.6	**Equation of a tangent to a circle**

A tangent is a straight line in the form $y = mx + c$. It touches a circle at one point.



In order to find the equation of a tangent, it is important to know that: $m_{tadius} \times m_{\tan gent} = -1$. This means that the radius and the tangent form a 90^0 angle at a point of contact.

Example:

Determine the equation of the tangent to the circle $(x-1)^2+(y+2)^2=10$ at the point (-2;-1).

Solution:

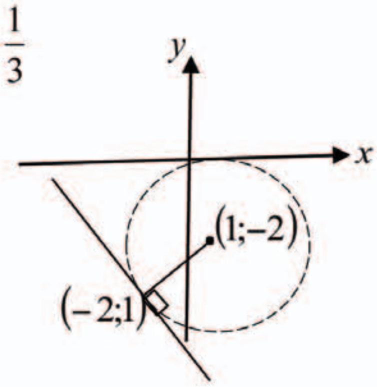
The co-ordinates of the centre of the circle as $(1;-2)$ and the other point $(-2;-1)$.

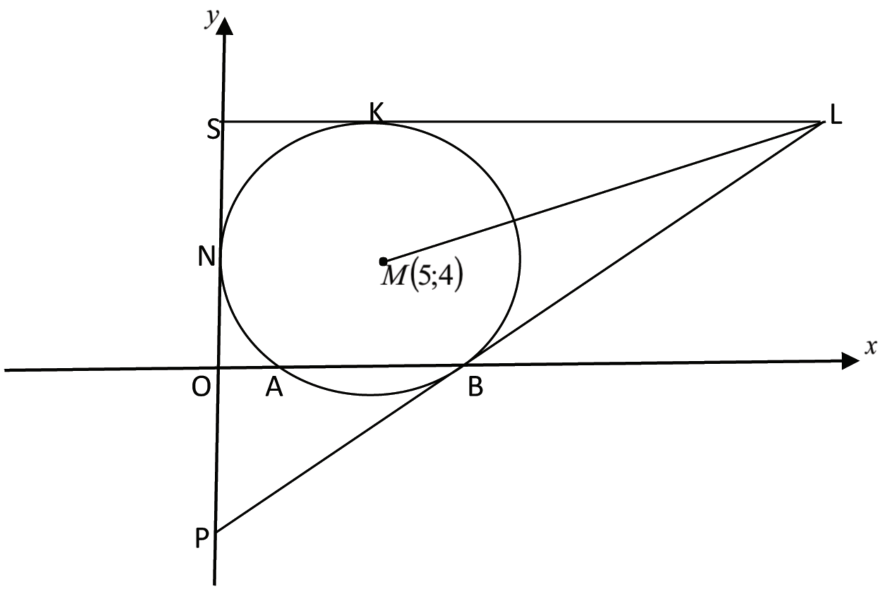
Calculate the gradient of the radius: $m_{radius} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-(-1)}{1-(-2)} = -\frac{1}{3}$

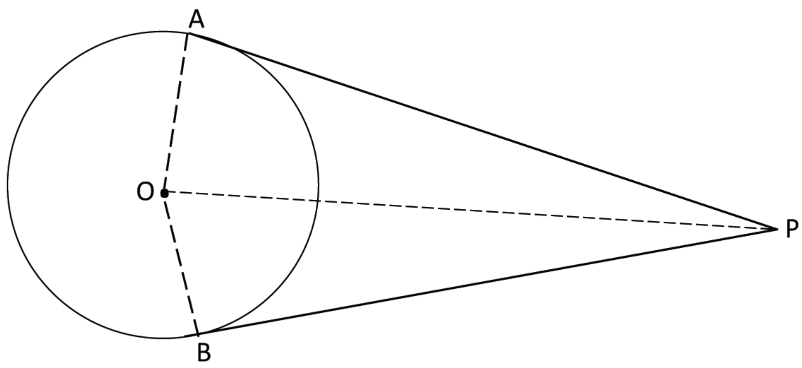
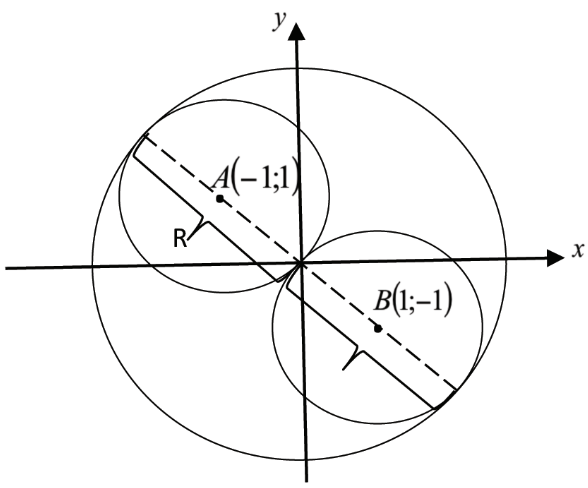
So, the gradient of the tangent is: $m_{radius} \times m_{tangent} = -1$

$$-\frac{1}{3} \times m_{tangent} = -1$$

$$m_{tangent} = -1 \div -\frac{1}{3} = 3$$



	The equation of the tangent: $y - y_1 = m(x - x_1)$ $y - (-1) = 3(x - (-2))$ $y + 1 = 3x + 6$ $y = 3x + 5$
	NSC 2014 Nov.
	In the diagram below, a circle with centre M (5;4) touches the y-axis at N and intersects the x-axis at A and B. PBL and SKL are tangents to the circle, where SKL is parallel to the x-axis and P and S are points on the y-axis. LM is drawn.
	
	Write down the length of the radius of the circle having centre M.
	Write down the equation of the circle having centre M in the form $(x-a)^2 + (y-b)^2 = r^2$
	Calculate the coordinates of A.
	If the coordinates of B are (8 ; 0), calculate: (a) The gradient of MB. (b) The equation of the tangent PB in the form $y = mx + c$.
	Write down the equation of the tangent SKL.
	Show that L is the point (20 ; 9).
	Calculate the length of ML in surd form.
	Determine the equation of the circle passing through points K, L and M in the form $(x-p)^2 + (y-q)^2 = c^2$

	Other impotant facts regarding circles and tangents: Tangents drawn from a common point are equal in length.  $PA = PB$ $\therefore [\Delta PAO \equiv \Delta PBO]$
	Given the circles $x^2 + y^2 + 2x - 6y + 9$ and $x^2 + y^2 - 4x - 6y + 9$, show that the two circles touch externally.
5.5.7	
	Consider the diagram above: The equation of the bigger circle is given by $x^2 + y^2 = 8$. The smaller circles are centred at A and B respectively. 3.1 Give the equations of circles centred at A and B respectively. 3.2 Prove that circle centred A and the bigger circle touch internally. 3.3 Give the new equation of the bigger circle, if it is translated 2 up and 3 left.

5 Check your answers

- Indicate answers / responses to the questions and activities in this section.
- Please ensure accurate correlation to the activity in the previous section.
- Indicate mark allocation (use ticks ✓) where required, but more importantly, explain how marks are allocated in the examination.

6 Message to Grade 12 learners from the writers

Mathematics can be fun, as it requires you to pull together all the pieces you learnt in the lower grades, in order to answer the Grade 12 examination. If you skipped one grade before Grade 12, it would leave a void in the grounding you require to pass the final examinations. Please ensure that you know all axioms and corollaries (all the rules) to answer the questions. Revise these by working through at least three sets of previous Mathematics question papers before you sit the final examinations.

Write one paper in 3 hours and mark your script on your own, using the memorandum, to gauge whether you are ready for the final paper. Memoranda for all previous DBE examinations are available on the DBE website.

We assure you that this year's final paper will be similar to those of previous years in both format and style.

Limit the amount of food you eat, in order for your cerebrum (brain) to work effectively. Digestion occupies the functioning of your brain.

Blast the final paper.

7 Thank you and acknowledgements

We hope the guidance provided in this booklet helps you in the final examination.
Mr Leonard Gumani Mudau, Mrs Mongameli Mbusi, Mr Muthige Ntshengedzeni Steven,
Mrs Nontobeko Tom and Mr Zulu Bhekani all wish you well.

TECHNICAL MATHEMATICS

ANALYTICAL GEOMETRY

GRADE 12

Department of Basic Education
222 Struben Street, Pretoria, 0001
Private Bag X895, Pretoria, 0001, South Africa
Tel: (012) 357 3000 Fax: (012) 323 0601

Hotline: 0800 202 933

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