MATHEMATICS TEACHING and LEARNING FRAMEWORK FOR SOUTH AFRICA

Teaching Mathematics For Understanding

basic education
Department:
Basic Education
REPUBLIC OF SOUTH AFRICA

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Teaching Mathematics For Understanding

DISCLAIMER

The Mathematics Teaching and Learning Framework for South Africa: *Teaching Mathematics for Understanding* does not replace the National Curriculum and Assessment Policy Statement (CAPS) for Mathematics.

If any editorial, layout, content, terminology or formulae inconsistencies are detected, the user is kindly requested to bring this to the attention of the Department of Basic Education.

FOREWORD BY THE MINISTER

During the 2016 Mathematics Indaba, I called for the overhauling of the South African pedagogical-content knowledge outlook in Mathematics. I said that we needed to reinvigorate the teaching of mathematics in its entirety – classroom learning practices, content, teaching, and assessments. I emphasised the urgent need to pay particular attention to the development of a new curriculum for initial teacher education, induction and continuing professional development.

This Mathematics Teaching and Learning Framework presented here is a first step towards achieving exactly that. The Framework seeks to and succeeds in laying a firm foundation for a new manner in which mathematics is taught thus changing the way it's learned.

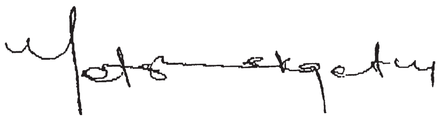
At the outset, I must emphasise that this Framework does not replace the Curriculum and Assessment Policy Statement (CAPS). Rather, it presents various options and new thinking about mathematics teaching, learning and assessments. The recommendations presented here represent the best international practise while taking into account our own unique South African needs and challenges. At the appropriate time, after due process of consultation, we will amend the CAPS accordingly, if so required. We welcome this initiative as it's an outcome of stakeholders' views on the role and place of mathematics in our basic education system.

The Mathematics Framework has far-reaching implications for the basic education sector. These implications will require the attention of teachers, education planners, and all other stakeholders if we are to be successful in implementing its recommendations. The Mathematics Framework's key takeaway is an ideal of, 'teaching of mathematics for understanding.' The teaching of mathematics for understanding in a learning-centred classroom methodology is designed to help teachers in both basic and higher education to attend to the challenges associated with the teaching and learning of mathematics, so that learner outcomes are improved.

The Mathematics Framework supports the key activities of the reviewed Mathematics, Science and Technology (MST) Education Strategy (2019-2030). The broad outline of the strategy is to ensure that every classroom is a space where quality learning and teaching takes place. This should be evident through the delivery of relevant curriculum content. The curriculum must be taught by competent and qualified teachers with the necessary resources to inspire learners with competencies for a changing world.

Finally, I would like to acknowledge the role of the Ministerial Task Team on Mathematics for the conceptualisation and development of this Framework. Equally, we extend a word of gratitude to all experts and stakeholders who participated in our very first Mathematics Indaba in 2016. To appoint the MTT on Mathematics was one of the rallying cries from the Indaba. Not only did we listen, but we have acted with the requisite speed.

It is with great pleasure that I present to you the Mathematics Teaching and Learning Framework – introducing a balanced and multi-dimensional approach for the teaching of Mathematics in South Africa.



MRS ANGELINA MATSIE MOTSHEKGA, MP

MINISTER

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LIST OF ACRONYMS

DBE:	Department of Basic Education
ANA:	Annual National Assessment
TIMSS:	Trends in Mathematics and Science Study
PISA:	Programme for International Student Assessment
SACMEQ:	Southern and Eastern Consortium for Monitoring Education Quality
LTSM:	Learning and Teaching Support Materials
ICT:	Information and Communication Technologies
NDP:	National Development Plan
CAPS:	Curriculum and Assessment Policy Statement
RNCS:	Revised National Curriculum Statement
GET:	General Education and Training (Grades R-9)
FET:	Further Education and Training (Grades 10-12)
FP:	Foundation Phase (Grades 1-3)
IP:	Intermediate Phase (Grades 4-6)
SP:	Senior Phase (Grades 7-9)
LoLT:	Language of learning and teaching
PANSALB:	Pan South African Language Board
UMALUSI:	Council for Quality Assurance in General and Further Education and Training

GLOSSARY OF TERMS

***Concepts**:* Mathematical concepts are the 'big ideas' of mathematics, such as number and operation concepts, measurement concepts (e.g. length, volume, capacity), algebraic concepts (e.g. variable, function, inverse) and so on. Knowledge of concepts enables learners to apply these ideas and, when appropriate, to identify algorithms to carry out mathematical procedures and to justify their mathematical thinking.

Procedures: The processes through which mathematics is done. Much of school mathematics involves procedural working which learners need to be able to perform fluently.

Strategies: The approaches used to do mathematical procedures and perform mathematical calculations. Learners should be able to use a variety of strategies and to devise their own strategies when they solve mathematical problems and do mathematical calculations.

Reasoning: Reasoning mathematically involves learners thinking and justifying their mathematical ideas. Two examples of mathematical reasoning are deductive and inductive reasoning. Mathematical communication is central to reasoning. Learners must learn to speak the language of mathematics for themselves.

Learning-centred classroom: A learning-centred classroom is characterised by a culture of interaction between teachers and learners, in the process of 'doing mathematics'. The teacher plays an important role in establishing this culture and in designing learning experiences to help learners learn mathematics, using whatever teaching and learning strategies s/he thinks are most suitable.

1+4 intervention: An intervention model (implemented in 2015) that attempted to address both subject content knowledge and methods of teaching strategies implemented in the school sector.

Number sense: A deep understanding of numbers that enables learners to work confidently and fluently with numbers because they have a sense of number structures and relationships.

Diagnostic assessment: A form of assessment that informs the teacher about learners' problem areas in a particular topic/s and which have the potential to hinder learner performance. Diagnostic assessment is not intended for promotion purposes; it is intended to diagnose the problems.

Formative assessment: Formative assessment is used to aid the teaching and learning processes, hence assessment ***for*** learning where learners get feedback on their performance in a way that helps them to improve.

Summative assessment: Summative assessment is carried out after the completion of a mathematics topic or a cluster of related topics. It is therefore referred to as assessment ***of*** learning since it is mainly focusing on the product of learning.

Manipulatives: Tangible materials used by teachers to exemplify concepts and to illustrate how procedures are done. For example, in the FP learners work with Unifix cubes and in the IP paper cut-outs can be used to promote the understanding of fractions.

***Home language (HL)**:* Policy uses this term to refer to the language that is spoken most frequently at home by a learner. This is also referred to as the 'main language', 'first language' or the 'mother tongue' of a learner in the literature.

***Language of learning and teaching (LoLT)**:* Refers to the language medium in which learning and teaching, including assessment, takes place. In South Africa this could be any of the 11 official languages, other languages approved by the Pan South African Language Board (PANSALB), Braille and South African Sign Language (SASL), approved by UMALUSI.

Translanguaging: Refers to a flexible use of language seen as an internal strategy by which speakers use all of their linguistic resources to communicate. ***Code switching**:* Refers to the conscious switching from one language to another language during teaching and learning.

EXECUTIVE SUMMARY

MATHEMATICS TEACHING AND LEARNING FRAMEWORK FOR SOUTH AFRICA:

TEACHING MATHEMATICS FOR UNDERSTANDING

The Mathematics Framework presented in this document, together with its recommendations for implementation calls for a multi-dimensional approach to transform the teaching and learning of Mathematics in South Africa. It calls for a balance in mathematics teaching between four key dimensions and proposes the implementation of this teaching in the context of a learning-centred classroom, where learners and teachers engage actively, discussing and experimenting with mathematical ideas.

The purpose of the Framework is to provide guidance to the mathematics education community in two ways. Firstly, it provides the theoretical background to the proposed balanced approach. Secondly, it includes worked exemplars that bring the dimensions of this balanced approach to life in the context of mathematical examples across all phases in the sector.

The five part Framework draws on Kilpatrick's et al's five strands of mathematical proficiency[1]. The Framework dimensions represent a contextualisation and adaptation of the strands to the South African context. The Framework proposes that steps should be taken to bring about the transformation of mathematics teaching in South Africa.

Teachers should strive to:

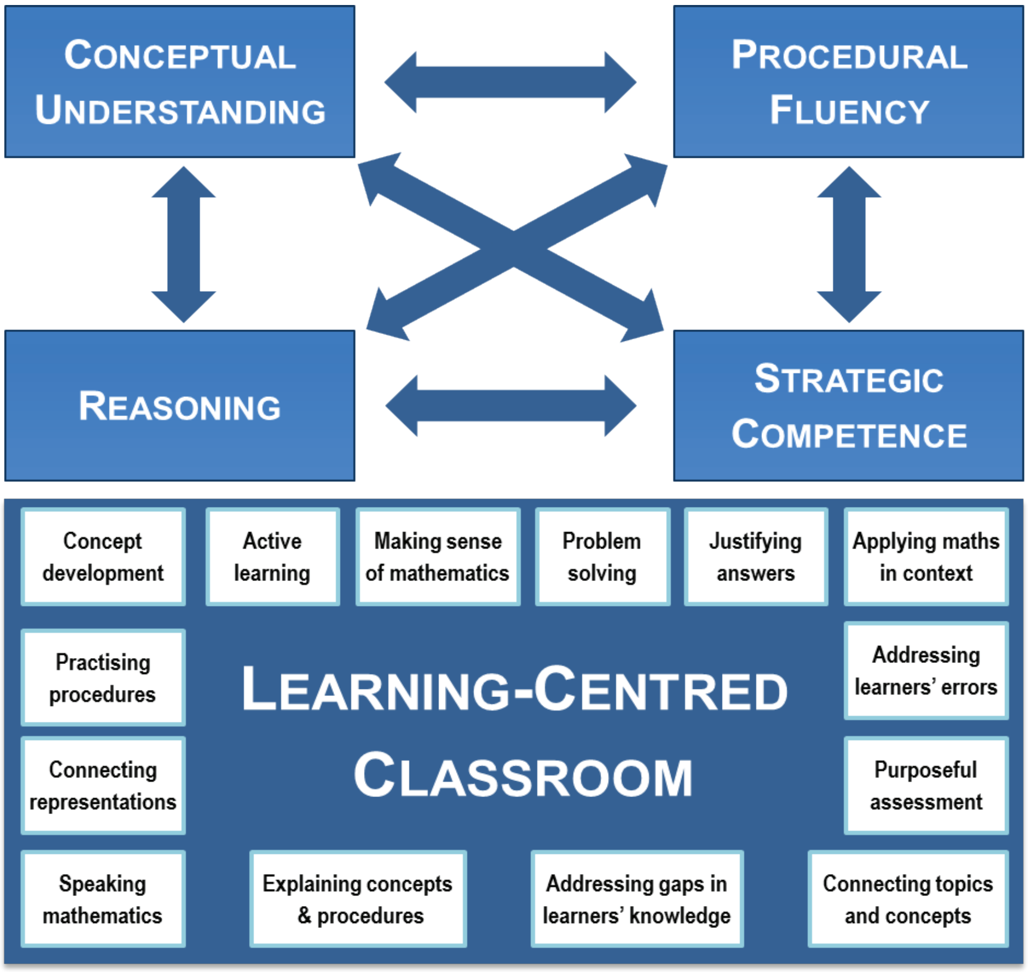
- teach mathematics for **conceptual understanding** to enable comprehension of mathematical concepts, operations, and relations;
- teach so that learners develop **procedural fluency** which involves skill in carrying out procedures flexibly, accurately, efficiently, and appropriately;
- develop learners' **strategic competence** – the ability to formulate, represent, and decide on appropriate strategies to solve mathematical problems;
- provide multiple and varied opportunities for learners to develop their mathematical **reasoning** skills – the capacity for logical thought, reflection, explanation, and justification; and
- promote a **learning-centred classroom** which enables all of the above, supported by teachers engaging with learners in ways that foreground mathematical learning for all.

The design of the Framework is simple and this document has been kept relatively short in order to make it more accessible to teachers. It is made up of four sections:

- An **introduction** to situate the Framework in the context of teaching and learning of mathematics in South Africa;
- A **theoretical exposition** of the Framework to outline and explain with examples, the reasoning behind the choice of the model;
- **Phase exemplars** to inspire teachers and offer guidance on implementing the four dimensions in a learning-centred classroom; and
- **Implications** of the Framework in the key educational areas of curriculum, assessment, teacher development (pre-service and in-service), Learning and Teaching Support Materials (LTSM), Information and Communication Technology (ICT) and Language of Learning and Teaching (LoLT).

1. Kilpatrick, J., Swafford, J. & Findell, B. (Ed.). (2001). *Adding It Up: Helping Children Learn Mathematics,* National Academy Press, Washington D.C.

The following diagram presents the Framework model of mathematics teaching and learning.



The dimensions of the Framework are all aimed at teaching mathematics for understanding.

Worked exemplars of the Framework dimensions are given in the text. Several (although not all) characteristics of a learning-centred classroom are shown in the diagram.

The Framework dimensions are characterised in the following way:

- **Conceptual understanding**

  Conceptual understanding enables learners to see mathematics as a connected web of concepts. They should be able to explain the relationships between different concepts and make links between concepts and related procedures. Conceptual knowledge enables learners to apply ideas and justify their thinking.

- **Procedural fluency**

  These are the processes through which mathematics is done. Learners need to perform mathematical procedures accurately and efficiently. They also need to know when to use a particular procedure.

- **Strategic competence**

  Learners should be able to identify and use appropriate strategies and to devise their own strategies to solve mathematical problems.

- **Reasoning**

  Reasoning includes justifying and explaining one's mathematical ideas, and communicating them using mathematical language and symbols. Mathematical reasoning includes deductive and inductive reasoning processes.

- **Learning-centred classroom**

  A learn**ing**-centred classroom focuses on *learning* – where the teacher designs learning experiences to help learners learn mathematics, using whatever teaching and learning strategies s/he thinks are most suitable for the specific lesson that will be taught.

The exemplars presented in the Framework are phase specific and draw on a range of content topics from the CAPS. They include practical examples for teachers relating to each dimension, tips for teaching and learning and a discussion of learning benefits associated with each dimension. These exemplars should be used by teachers to deepen their understanding of the Framework in order to apply the Framework dimensions to all of their teaching of Mathematics.

Brief notes on the implications of the Framework are discussed under the following headings:

- **Curriculum**: The Framework is not a new curriculum and does not replace CAPS. Issues such as scope, depth, sequencing and time allocation need to be reviewed.
- **Assessment:** A greater focus should be placed on formative assessment (assessment *for* learning). This should include the use of data to inform teaching such as using error analysis and to facilitate meaningful engagement with learner responses.
- **Learning and teaching support materials (LTSM) and Information and communication technology (ICT):** Fine-tuning of LTSM to reflect the Framework. Use of appropriate manipulatives and various technologies is advocated.
- **Teacher development (pre-service and in-service):** Intensive and systematic teacher development programmes should be set in motion to develop teachers' confidence in teaching mathematics for understanding.
- **Language of learning and teaching (LoLT):** Recognise the multilingual context. Recommend bilingual materials particularly in Grades R-4 and possibly in Grade 4-6.

This Framework is **not a new curriculum** and does not replace the existing curriculum. Instead it supports the implementation of the current curriculum through introducing a model to help teachers to change the way in which they teach. The Framework model and the supporting exemplars are provided to offer guidance to teachers that will enable them to transform their teaching. This transformation should lead to teaching for understanding, so that learning for understanding will take place in all mathematics classrooms in South Africa.

1. INTRODUCTION AND BACKGROUND

The teaching and learning of Mathematics in South African schools is not yielding the intended outcomes of South Africa's education policies and curricula. This is evident from research from many studies conducted by the Department of Basic Education (DBE), universities and other research agencies in South Africa. The low learner achievement levels revealed by national assessments such as Annual National Assessments (ANA), regional assessments such as Southern and Eastern Consortium for Monitoring Education Quality (SACMEQ) and international assessments such as Trends in Mathematics and Science Study (TIMSS) are indicative, at least in part, of current 'ineffective' teaching and learning practices.

Although South Africa's achievement levels in recent SACMEQ and TIMSS studies have shown some improvements (there has been a decrease in the percentages of learners who achieve at the lower mathematics levels of the SACMEQ hierarchies and an increase in the percentages that achieve higher levels) the scores are not yet adequate across the entire school system. It is evident that the numerous interventions implemented by the Education Sector such as developing and providing learners and teachers with good quality textbooks, the radical 1+4 Intervention Model that advocates professional learning communities, and self-study guides are not changing the country's mathematics performance significantly.

In the FET band, despite the vastly improved participation rate and slight improvements in learner achievement in Mathematics in the 2016 Grade 12 examinations, learner achievement has begun to plateau. Too many students struggle with passing the subject and on the top end, even students who perform well struggle at university. At this rate, the achievement of the National Development Plan (NDP) goal of 50% of learners and schools performing at 90%, will be difficult.

The NDP envisions that by 2030 schools will provide all learners with quality education, especially in Literacy, Mathematics and Science. To this effect, the acceptable level of performance to be achieved by the year 2030 has been determined and set to at least 50% and the target of learners and schools performing at this level has also been determined and set at 80%. Performance targets for Grade 6 and Grade 8 have respectively been set at 600 and 500 in SACMEQ and TIMSS respectively. Further, the NDP proposes that 450 000 learners should be eligible to study Mathematics at university by 2030.

A ground-breaking and sustainable intervention that will change the approach to teaching Mathematics is required to achieve these targets – a new outlook that informs teaching is needed if teachers are to change the way in which they present mathematics and engage with learners in their classes. This outlook should be underpinned by curriculum requirements and pedagogy – the way in which this curriculum is taught.

There are lessons to be learned from international best practices although the contexts within which teaching and learning occur in these countries should be remembered. The majority of countries, especially Asian countries, whose learners are performing well in Mathematics, have a specific pedagogic-content outlook that informs how teaching and learning of the subject should be approached, how textbooks should be written and how assessment should be conducted. Examples of this are the 'problem-solving approach' adopted by Singapore and the Realistic Mathematics Education tradition in Netherlands where Mathematics is viewed as 'human activity'. Japanese lesson study combines teacher development with curriculum planning very effectively.

For South Africa, the Curriculum and Assessment Policy Statement (CAPS) defines Mathematics thus:

> Mathematics is a **language** that makes use of symbols and notations to *describe numerical, geometric and graphical relationships*. It is a **human activity** that involves observing, representing and investigating patterns and quantitative relationships in physical and social phenomena and between mathematical objects themselves. It helps to **develop mental processes** that enhance *logical and critical thinking, accuracy and problem-solving* that will contribute in decision-making.

This definition of Mathematics provides some cues in terms of the direction of the 'new' pedagogic-content outlook. Mathematics teachers should be planning and presenting lessons that engage learners in conceptual thinking about mathematical ideas, developing their mathematical language in order to express themselves mathematically, building their procedural competence in ways that enable them to use mathematical procedures effectively in both routine and problem solving activities.

2. PURPOSE OF THE FRAMEWORK

This document has been developed for South African teachers to guide and assist them to teach Mathematics in a way that improves learner outcomes. Reflection on the mathematical pedagogic-content orientations and the classroom contexts of several countries which perform well in mathematics international benchmark tests (such as TIMSS and PISA) reveals information that can be used as a guide in the formulation of necessary interventions to transform Mathematics teaching in South Africa. Based on lessons learnt from these countries, together with the context of the South African teaching and learning environment, this framework aims to provide what is envisaged to be the *South African Mathematics Teaching and Learning Identity*.

The document is divided in two parts:

Part 1 presents the Framework and exemplars; and

Part 2 focuses on possible implications of the Framework in relation to: Curriculum, Assessment, Learning and Teaching Support Materials (LTSM), Information and Communication Technologies (ICT), Teacher development (both pre- and in-service) and the Language of Learning and Teaching (LoLT).

3. TEACHING MATHEMATICS FOR UNDERSTANDING

Since the advent of a democratic South Africa in 1994, education became one of the important drivers through which the transformation agenda was advanced. To this end, education has been viewed as an apex priority of government and subsequently a societal priority. In aiming to ensure that the Right to Education, as enshrined in the Bill of Rights, is maintained and that quality education is provided to all South African learners, the country introduced three curriculum reviews since 1994. The first two, Curriculum 2005 and the Revised National Curriculum Statement (RNCS), both with strong outcomes-based underpinnings, and thirdly the Curriculum and Assessment Policy Statement (CAPS) which is mainly content-driven. Despite the good intentions embodied in the curriculum changes, South Arica is still grappling to improve learner performance in Mathematics.

This framework is **not a new curriculum** and does not replace the CAPS. Instead it supports the implementation of the current curriculum through introducing a model to help teachers to change the way in which they teach. The model has four dimensions which are exemplified in the framework document to offer guidance to teachers that will enable them to transform their teaching. This transformation should lead to teaching for understanding, so that in all South African mathematics classrooms, there is learning for understanding taking place.

3.1 Mathematical curriculum underpinnings

The current South African curriculum statement, the CAPS, is comparable in quality, breadth and depth to mathematics curricula of other countries world-wide. If it were to be implemented as it was conceptualised, the CAPS has the potential to, *firstly* equip the South African learners with the skills for the 21st Century and, *secondly* prepare them adequately for the demands of the 4^{th} Industrial Revolution which emphasises cyber-physical production systems as espoused by the World Economic Forum. Remarkably a significant number of skills espoused by the 21^{st} Century Skills and those that are needed to thrive in the job market of the 4^{th} Industrial Revolution era feature quite prominently in the CAPS as part of the general aims of the curriculum. The skills include, but are not limited to, problem-solving, mathematical reasoning, logical reasoning, cognitive flexibility and ICT literacy.

There seems to be a disjuncture between the concerning situation presented above (poor learner outcomes in mathematics) and the implementation of the research informed, progressive and dynamic mathematics curriculum (CAPS) in the majority of schools in South Africa. It is to address this disjuncture that the Mathematics Framework has been developed – to assist teachers to pay attention to key features within this curriculum that will enable them to take the necessary steps to transform the manner in which mathematics is taught (and hence learned). The five part framework presented here has been influenced by the conceptualisation of Kilpatrick's five strands of mathematical proficiency[2]. The framework dimensions represent a contextualisation and adaptation of the strands to the South African context. The steps that should be taken to bring about the transformation of mathematics teaching in South Africa involve the following. Teachers should strive to:

- teach mathematics for **conceptual understanding** to enable comprehension of mathematical concepts, operations, and relations;
- teach so that learners develop **procedural fluency** which involves skill in carrying out procedures flexibly, accurately, efficiently, and appropriately;
- develop learners' **strategic competence – the** ability to formulate, represent, and decide on appropriate strategies to solve mathematical problems;
- provide multiple and varied opportunities for learners to develop their mathematical **reasoning** skills – the capacity for logical thought, reflection, explanation, and justification; and
- promote a **learning-centred classroom** which enables all of the above, supported by teachers engaging with learners in ways that foreground mathematical learning for all.

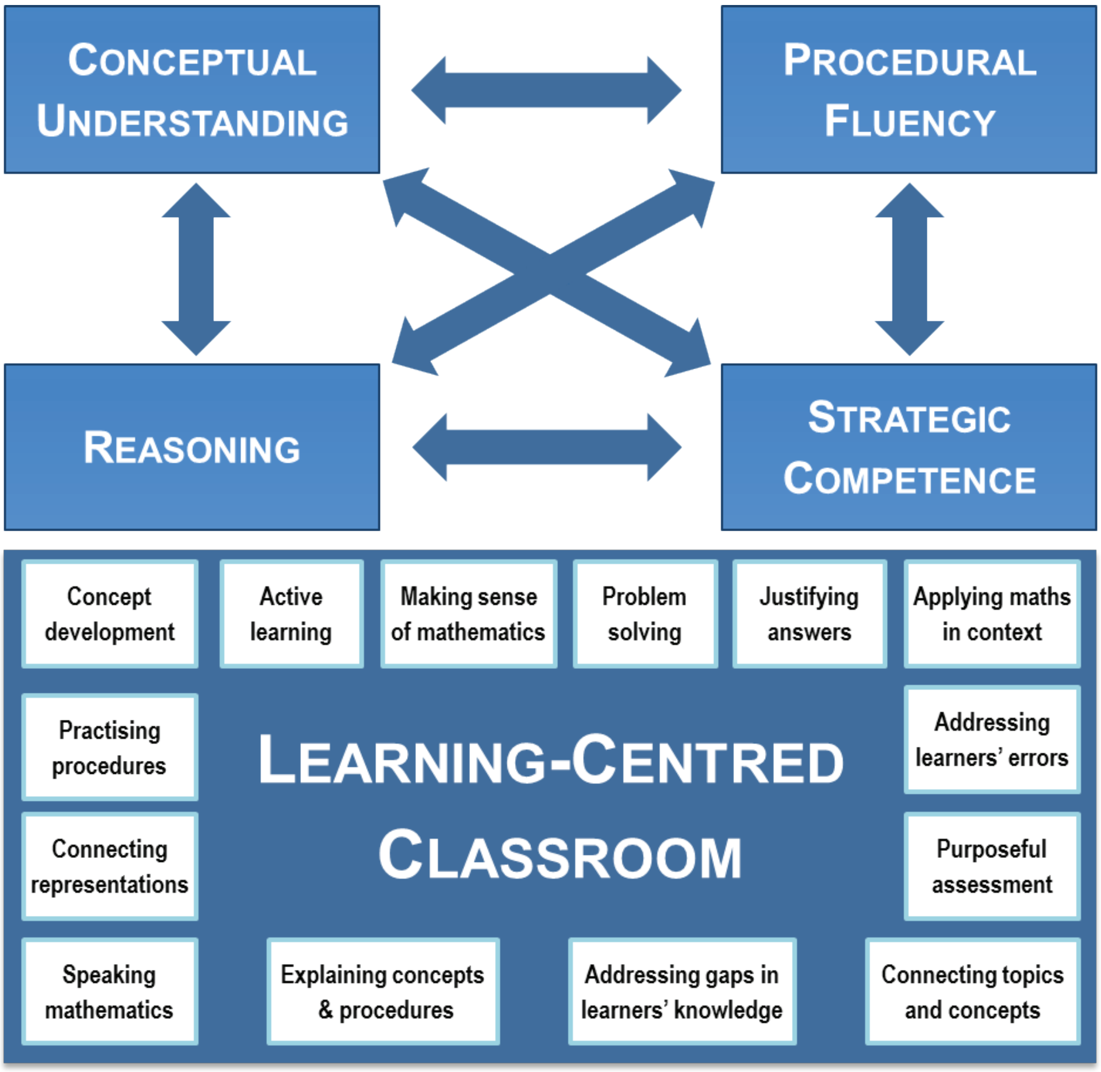
3.2 The proposed model of mathematics teaching and learning

The model of mathematics teaching and learning presented in this framework is constituted by four key dimensions: conceptual understanding, mathematics procedures, strategic competence, and reasoning; while each of these is underpinned by a learning-centred classroom. Although the four key dimensions are interdependent and should be properly linked to optimise effective teaching and learning of mathematics, it could be argued that more emphasis should be placed on conceptual understanding since this is the metaphorical foundation on which all other dimensions build. The emphasis on conceptual understanding is a purposeful move to address the common teaching and learning practice which is characterised by memorisation of mathematical procedures with little understanding of how they were derived, why they work and when they are relevant.

The framework model of mathematics teaching and learning is illustrated diagrammatically in Figure 1. This is followed by an example which unpacks the theory in a mathematical context. The example is based on Foundation Phase level subtraction (63 – 49 =). Other examples across all phases are provided later in the framework.

2. Kilpatrick, K., Swafford, J. & Findell, B. (Ed.). (2001). *Adding It Up Helping Children Learn*. Mathematics National Academy Press, Washington D.C.

Figure 1: Model of mathematics teaching and learning.



Several (although not all) characteristics of a learning-centred classroom are given in **Figure 1.**

The example discussed below exemplifies the four dimensions in relation to addition and subtraction of 2-digit numbers.

Conceptual understanding: Key to place value is being able to work flexibly with composite units – treating 10 as either 'one unit of ten' or 'ten units of one'.

Procedural fluency: Adding or subtracting 10 to any 2-digit number (without counting in ones), adding a single digit to any two digit number (extending the number bonds).

Strategic competence:

- Using jumps of 10s and units: 63 – 49. "Start at sixty-three, jump back forty, that gives twenty-three. To jump back nine, jump back three to get to the ten before, which is twenty and then jump back the remaining six to land at 14".
- Using compensation: 63 – 49. "Sixty-three minus fifty is twenty-three. That's subtracting one more than I need to, so I have to add one back. The answer is 14".
- Using place value blocks: Make sixty three as '6 tens and 3 units'. We don't have enough units here to take away the nine units in 49, so exchange one of the ten strips for ten units, leaving us with 5 tens and 13 units. We can now subtract 4 tens and 9 units, leaving 1 ten and 4 units = 14".

Reasoning: Given that 63 – 50 = 13 learners can reason that 63 – 49 must be 14 and explain that 'if you subtract 49 from 63, that is taking away one less than 50 so the answer must be one more. There are various elements of reasoning across the strategies described above.

3.3 Exposition of the model

Each dimension of the framework model is explained below followed by an expansion of the mathematical examples shown in Figure 2. The dimensions can be focused on individually but they are all interconnected – this should not be forgotten. Note that the use of mathematical procedures, reasoning and learners' own methods all build conceptual understanding – in this case of number and place value. The concrete demonstration with the blocks lays the foundational understanding for why numbers can be broken up in different ways to do calculations using different procedures and strategies. Knowledge of place value enables learners to break up the number 63 into 6 tens and 3 units and 49 into 4 tens and 9 units, to perform certain calculations. Number sense allows learners to work flexibly with numbers and to know that numbers can be broken up in different ways to enable subtraction.

Modern societies and economies are in a constant state of flux. It is no longer sufficient for learners only to learn how to reproduce the steps in the calculations that they are shown by teachers. It is also a function of the nature of mathematics as connected and hierarchical that yields memory as insufficient for progression in mathematics. If we are to equip learners for the future, they need to be adaptable. They must use knowledge and skills in flexible ways. Mathematics teaching and learning should develop learners who can adapt their knowledge and use their skills flexibly, not only as the basis for solving problems, but also as the foundation for learning new skills and knowledge.

(a) Conceptual understanding

Conceptual knowledge is knowledge of concepts, relations, and patterns. It assists and enables learners to make sense of mathematics. Learners who have a sound grasp of conceptual knowledge, when asked to justify their work, would not say, 'My teacher told me to do it like this' rather they will be able to explain the reasoning behind their work.

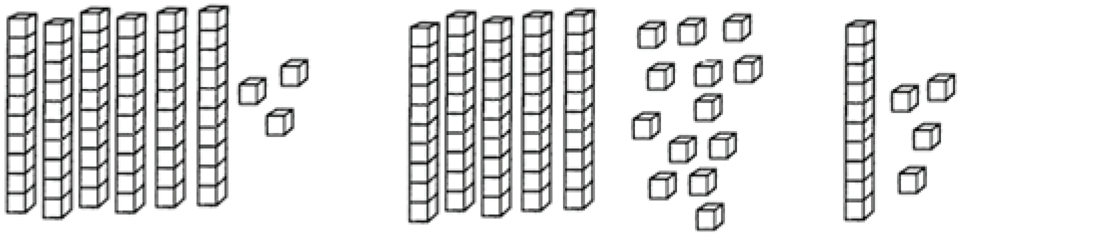
Learners who have conceptual knowledge are able to compare, relate and infer. They can make connections between ideas. Higher level thinking is fundamental to conceptual knowledge. Conceptual knowledge is constructed as problems are solved, investigations are carried out, and questions are pondered. Meanings and connections among ideas develop as learners work with concrete, pictorial, and symbolic material, as they reflect on what they have done, and as they communicate with others. However, learners often need teachers to thoughtfully and strategically push them to progress from concrete, specific working to more abstract generalised ideas. The more the learners are exposed to such ways of working, the better the chances become that they will develop into mathematical problem solvers.

When a new concept is introduced to learners, the teacher should plan a lesson for conceptual development. This process places considerable pedagogic demands on the teacher. All introductory lessons should be carefully planned. Learners could be engaged by sharing their ways of thinking and the teacher should use these ideas to challenge learners to construct and connect ideas through reflective discussions on the methods.

Teachers might choose to use appropriate manipulatives and bring in aspects of learners' out of school experiences in order to help learners to build their knowledge.

For example, a learner finding the difference between two 2-digit numbers could be shown how the calculation works using base ten blocks. The concept of subtraction, which was introduced in Grade 1 is reinforced in this Grade 2 example, by the concrete demonstration, which can be linked to a numeric calculation to help the learner make the connection between concrete working and abstract calculations.

63 – 49 =



Lay out 63 using base ten blocks | Exchange one ten for ten ones | 14 remains after subtracing 49

I have 6 tens and 3 ones from which I must subtract 49.	I have 5 tens and 13 ones after I exchange in order to be able to subtract.	If I take away 4 tens and 9 ones, I have 1 ten and 4 ones left, that is 14 left.

The concrete working with base ten blocks shown above demonstrates the exchange needed in order to do the subtraction. This concrete activity builds up learners' conceptual understanding of number and operations. The written record of the procedure is based on conceptual knowledge. Learners need connections to be made explicit in order to become fluent in the use of procedures.

(b) Procedural fluency

Whereas conceptual understanding is an implicit or explicit understanding of the interrelations between pieces of knowledge, procedural knowledge is seen as the sequence of actions that are performed to solve a problem. These two types of knowledge do not develop independently. Conceptual understanding often leads into the procedures that a learner will use. Often conceptual understanding precedes procedural skills. A teacher who is aware of the importance of conceptual understanding when teaching concepts will not teach the procedural skills before learners have mastered the concepts involved.

Procedural knowledge is the recognition of symbols and the ability to follow rules to 'do' mathematics. It can be thought of as having mathematical skills and carrying out actions in a correct sequence. Mathematical expertise involves both conceptual and procedural knowledge and also awareness that procedures are based on mathematical principles. Conceptual and procedural knowledge support each other and work together to attain mathematical power. Procedures connected to conceptual knowledge give flexibility to mathematical thinking and enable learners to extend the range of both types of knowledge when new problems arise. If children learn procedures without understanding, their knowledge may be limited to meaningless routines.

Research has shown that increasing learners' conceptual knowledge leads to the ability to generate one's own procedures. There is a reciprocal relationship between conceptual and procedural knowledge, but as argued above conceptual knowledge has a stronger and more foundational role to play in developing procedural knowledge than the reverse. Conceptual understanding leads to the generation of flexible procedures, and procedural knowledge can lead to conceptual understanding. Similarly conceptual teaching (teaching focused on developing conceptual understanding) enables the effective teaching of flexible procedures and these procedures then enables strengthened engagement with conceptual understanding in one's teaching.

Developing procedural fluencies for adding and subtracting numbers is essential for further mathematical learning. Fluency is developed through much repetition and practice.

Adding or subtracting 10 to any 2-digit number (without counting in ones), adding a single digit to any two digit number.

(c) Strategic competence

Learners should be able to make sensible decisions on what strategies to employ or to devise their own strategies to solve certain problems. Often there is more than one way to solve a mathematical problem and it is important that learners do not always depend on fixed, prescribed methods to solve problems. This dimension includes two (related) skills:

a) *Strategic competence* - the ability to formulate, represent and solve mathematical problems. Learners should be able to read and make sense of a mathematical problem, look for possible patterns and use some strategy to solve the problem. There may be a variety of strategies that are useful in different contexts. Part of strategic competence is the ability to select and use an appropriate strategy in a given context.

b) Learners using their *own strategies* to approach a problem that cannot be solved using familiar strategies. We tend to focus on technical, procedural aspects of mathematics and learners do not get the much needed exposure to *problem solving* which is an integral part of mathematics. Learners need this exposure in order to develop their capacity to generate their own strategies since this is a basis for problem solving. Learners need to have and be able to use procedures which they have at their disposal.

It is particularly useful to allow learners to use their own strategies when introducing a topic or a new section of a topic, because it can help them to develop a better conceptual understanding of the topic. When learners discuss and compare their own strategies they can learn ways of reasoning and improving their calculations from their peers. As learners try out new strategies they can find out whether the initial strategy used is particular to the example, or whether it is generalisable to many examples.

For example, learners could use many different strategies to calculate subtraction. Two examples of strategies are shown below. There are many other possibilities! The strategies below should not be taught in a rote fashion as this can lead to misconceptions or incorrect use of the procedure. Correct working with mathematical strategies builds on and develops learners' number sense but only if learners understand how the procedure works.

63 – 49 =

Strategy 1:

63 – 49 = 63 – 40 – 9 (the learner broke down 40 into 40 and 9 …

= 23 – 9

= 14

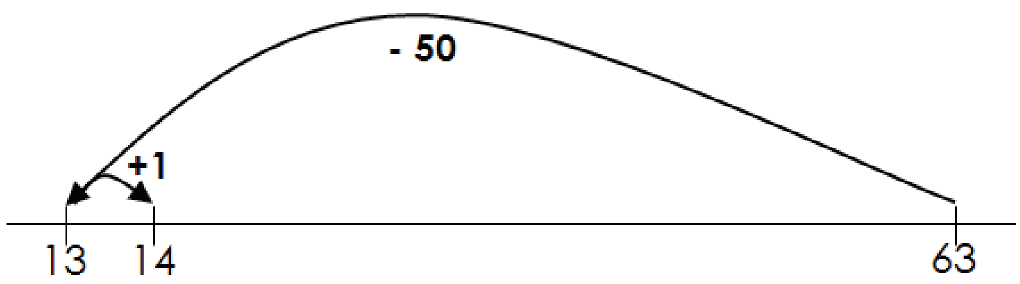
Strategy 2:

63 – 49 = 63 – 50 then add 1 back

so 13 plus 1

so the answer is 14

This can be shown on a number line:



Teachers could also discuss with learners how and why the vertical algorithm works with addition and subtraction (additive relations) if this strategy is used by learners. This might occur because parents or siblings or other individuals might take time to do mathematics with learners outside of the classroom. The vertical (column) method of recording numeric calculations links with the structure of place value, thus it can (and should) be explained using place value. It is an efficient and effective strategy that is used widely around the world. This strategy could be shown to learners along with the other strategies for adding and subtracting numbers.

Ultimately, learners should be guided to choose which strategy they find most efficient and prefer. They should also learn how to choose which strategy is most appropriate in a given context.

c) Learners need to develop their ability to *think out of the box* (i.e. to find strategies that have not been shown to them before) since this is useful for effective problem solving in mathematics. Bearing this in mind, the framework provides an example per phase of a more 'open' problem in addition to the more standard curriculum linked examples. When learners 'think out of the box' they are applying mathematical reasoning.

The beginning of 'out of the box' thinking is when learners are able to act independently – moving towards fluency and efficiency in the use of procedures in the solution of problems. An effective way of giving learners exposure to problem solving is through participation in mathematics competitions. These competitions include questions that learners have not generally seen before and for which standard taught procedures will not be sufficient to solve the problems. The perception that 'mathematics competitions are only for the gifted learner' is a myth. All learners, not only gifted learners, need to develop the thinking skills needed to solve non-standard problems. There are two official national events, the SA Mathematics Challenge (SAMC) for primary school learners, and the SA Mathematics Olympiad (SAMO) for secondary school learners (both organised by the SA Mathematics Foundation and endorsed by the national Department of Basic Education)[3]. There are also other regional or local mathematics competitions. Participating in these competitions could enable learners to grow their mathematical thinking skills.

(d) Reasoning

Many learners see mathematics as a system of algorithms to be performed to get 'the right' answer. Along with other components such as creativity and intuition, logic forms an integral part of mathematical thinking, however even adults sometimes find it difficult to reason in a formal logical way. To help learners reason mathematically, we need to teach them skills they do not possess naturally.

Mathematics is not simply a collection of isolated procedures and facts; it consists of a web of interconnected concepts and relationships. If learners are taught mathematics as a series of disconnected procedures that need to be learnt off by heart, they are likely to experience mathematics as meaningless. It will also mean that they have more to memorise which deprives them of the opportunity to develop higher order thinking skills. If, on the other hand, learners are encouraged to connect topics and develop the practice of thinking '*What do I already know, that can help me here?*', they will reduce the strain on their memory and increase their reasoning abilities.

Learners use *inductive reasoning* to make generalisations based on evidence they have found. In this kind of reasoning facts are usually accumulated to convince us of the validity of a particular statement. *Inductive reasoning* starts with specific examples or observations and leads to a conjecture about the apparent rules or patterns that lie behind them. A mathematical example of *inductive reasoning* is the identification of a pattern in a sequence of numbers.

A large part of mathematics is based on an axiomatic system in which *deductive reasoning* is the accepted route to gain new mathematical knowledge. *Deductive reasoning* starts with the rules (or axioms), and through *deductive reasoning*, we determine what the consequences will be. This is what mathematicians do in most of mathematics, defining the rules for a mathematical entity (such as the basic axioms for Euclidean Geometry), and using these rules to prove that other, more complicated, facts are true. With *deductive reasoning* we can be absolutely sure of our conclusions - as long as we assume the axioms are true.

3. Learners' from all Quintile 1 and 2 schools participate in these events without ANY entrance fee - that is for both the SAMC and the SAMO. For other schools there is a small entry fee but the DBE makes available funding to all MST schools to pay for their entry fees.

Logical deductive reasoning is an important foundational skill in mathematics. Learning mathematics is a sequential process of building connections. If learners do not fully grasp a certain concept or procedure, they may struggle to understand other concepts or procedures that follow – since these may depend on or build on the earlier ideas. Disciplined deductive mathematical reasoning is crucial to understanding and to using mathematics properly.

Reasoning mathematically involves learners talking about mathematics. Learners must learn to speak the language of mathematics for themselves. They cannot do this without being given opportunities to 'talk mathematics'. Teachers should support learners as they learn to develop their mathematical language. At times learners may be able to get answers but struggle to explain how they got to the answers. Teachers should support learners to develop the language and skills needed to talk about their thinking, answers and solution strategies.

As learners progress they need to learn to work (and speak about mathematical objects) more abstractly. This requires them to start to reason by making use of formal mathematical definitions in order to justify their answer or to build an argument.

For example, learners need to move from a simplistic claim that a shape is a rectangle: 'it is a rectangle because it's a longish shape with two long sides and two short sides and four corners' to a more sophisticated argument: 'a shape is a rectangle if its opposite sides are equal **and** it has four right angles'. In the first case learners will not acknowledge that a square is also a rectangle while in the more formal abstract discourse a square is clearly a special case of a rectangle.

In the example below the Grade 2 learner's answer shows the use of mathematical language to explain how she reasoned when finding a solution based on given information.

Given that 63 – 50 = 13 learners can reason that 63 – 49 must be 14 and explain that 'if you subtract 49 from 63, that is taking away one less than 50 so the answer must be one more. There are various elements of reasoning across the strategies described above.

(e) Learning-centred classroom

A learning-centred classroom creates a platform for meaningful learning and teaching. The framework diagram illustrates this by placing the learning-centred classroom as the foundation for all of the other dimensions of the framework. Teachers need to create classrooms where the stage is set for learning mathematics for understanding. The term "learning-centred" has been chosen very deliberately. Much has been written and said about *learner-centred* and *teacher-centred* classrooms. Often it has been suggested that "teacher-centred is bad" and "learner-centred is good". Such dichotomies are not helpful and they are not accurate. There are many reports of so-called learner-centred classrooms where it was not clear what learners were supposed to be learning. On the other hand, we read of so-called teacher-centred teaching where learners displayed a good grasp of what their teacher had explained using a so-called chalk-and-talk approach.

A **learning**-centred classroom focuses on *learning* – where the teacher designs learning experiences to help learners learn mathematics, using whatever teaching and learning strategies s/he thinks are most suitable for the specific lesson that will be taught.

A **learning**-centred mathematics classroom is characterised by a culture of interaction between teachers and learners, in the process of 'doing mathematics'. The teacher plays an important role in establishing and nurturing this culture. The way in which a teacher conducts a classroom, depends on the way in which s/he views mathematics. A teacher, who sees mathematics as a body of knowledge which s/he has to impart to learners, will mostly tell learners what to do, and how to do it. On the other hand, a teacher who sees mathematics as a body of knowledge that learners must actively explore and engage with, will create a learning environment where learners can make sense of mathematics. They will have opportunity to express their ideas, to ask questions of the teacher and each other and discuss their ways of thinking about the work at hand.

Teachers must direct and be in control of the path of learning but they must see the learners' role as active in developing understanding and taking ownership of what they have learned. For example, in the Foundation Phase, focused and guided play-based learning should be promoted because it is an important component of active learning of mathematics.

The CAPS Grades R - 12 aims to develop citizens that are able to:

- identify and solve problems and make decisions using critical and creative thinking;
- work effectively as individuals and with others as members of a team;
- organise and manage themselves and their activities responsibly and effectively;
- collect, analyse, organise and critically evaluate information;
- communicate effectively using visual, symbolic and/or language skills in various modes;
- use science and technology effectively and critically showing responsibility towards the environment and the health of others; and
- demonstrate an understanding of the world as a set of related systems by recognising that problem solving contexts do not exist in isolation.

It is evident that the CAPS emphasises the importance of developing learners' ability to solve problems and to share and communicate their ideas. In a **learning**-centred mathematics classroom learners will have opportunity to:

- make sense of mathematics
- speak mathematics
- develop fluency in essential mathematical procedures
- connect representations
- justify their thinking

In a **learning**-centred mathematics classroom teachers will:

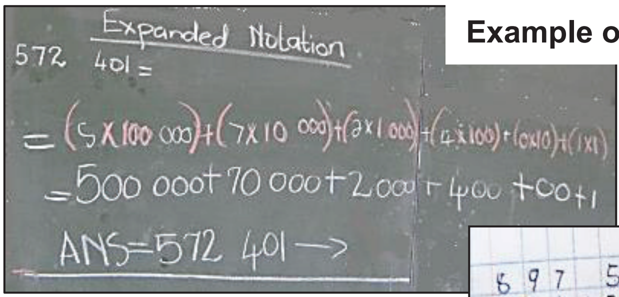
- use assessment for learning
- provide clear explanations of concepts and procedures
- address learners' errors and misconceptions
- address gaps in learners' knowledge
- make connections between different topics
- provide opportunities for active learning
- select and design tasks that emphasise key mathematical ideas and ways of working mathematically
- encourage learners to speak mathematics and use mathematical notation accurately

All four dimensions of the framework are closely linked and in a learning-centred classroom they will interact dynamically. Some dimensions will come to the foreground in some lessons while other dimensions will come more strongly into focus in other lessons. All this comes together as teachers strive to teach mathematics for understanding to enable learners to learn mathematics with understanding.

Let us take a look at two scenarios in an Intermediate Phase classroom:

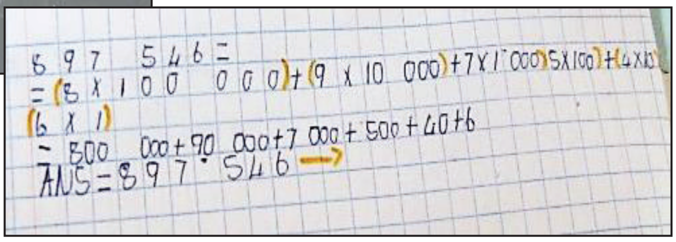
Example 1: Writing numbers using expanded notation

In the first example, the teacher does an example on the chalk board and learners then copy this example into their workbooks. Thereafter, they do several similar examples in their books.



Example on chalk board

Example in workbook



In the chalk board shown above, the teacher has provided a clear example of a suitable layout for expanded notation. Learners are then able to copy this into their books and can refer to it for future examples, as can be seen in the learner's workbook for the example 897 546.

Example 2: Finding the formula for volume of a rectangular prism

In the second example, Grade 6 learners work towards finding the formula for calculating the volume of a rectangular prism. The teacher guides the learners by asking questions that provoke the learners to draw on their understanding of the concept of volume. For example: 'So when you look at that rectangular tower or prism of mini blocks, how did you calculate how many mini blocks there were in each tower?' Here the teacher is pushing the learners to generalise the strategies they use to find the volume in cubed units. They do this by finding the volume of various rectangular prisms.

The language used by learners in this discussion might need to be refined, supported by the teacher – it could progress towards abstraction in the following steps:

- The volume is the total number of blocks. I can find it by counting the blocks. So many layers of blocks, with so many blocks in a layer …
- The number of bocks in one layer times the number of layers …
- The number of blocks in one layer (the base) is number of blocks in the length times number of blocks in the breadth of the base. The number of layers gives me the height. The volume is the total number of blocks: number of blocks in the base times height.
- Volume (of a rectangular prism) = length x breadth x height (and it is a number of blocks or cubes).
- V = l x b x h (measured in cubic units)

An interactive lesson can develop around examples which a teacher has written on the board or in the context of another activity planned by the teacher.

Teaching in an environment where learners are active and talk through their own learning calls for careful teacher preparation. Teachers might have to take on a different role, and that could make them feel insecure. A learning-centred classroom calls for more freedom on the part of learners. This requires careful planning from the teacher.

The teacher must prepare the lessons in much more detail, taking into consideration the possible questions and problems that learners could encounter. As teachers work consistently towards creating a learning-centred classroom environment, they will become more familiar with the type of questions and problems that learners encounter.

Teachers will be better prepared to teach interactively as they become more experienced in creating a learning-centred classroom. Once the initial work has been done to create scenarios where learners are involved in solving problems collaboratively and independently, the teaching load will become less.

Teachers are sometimes under the impression that they must control of all of the learning that takes place in their classrooms when in fact this is not the case. Teachers should be in control of the learning pathways they direct, but they must do this to create independent learners.

Learners must take responsibility for and control of their own learning in order to become independent learners and thinkers, able to operate mathematically in the world without the teacher.

The role of the teacher should thus be to create an environment in which learners are provided rich activities through which they can develop their independence and control their own learning. To create such a classroom is not always easy but the benefits to be reaped are those mentioned above – independent learners, capable of thinking and working on their own and doing mathematics in a meaningful way.

3.4 Phase exemplars of the dimensions of the model

The four dimensions of the framework apply to all mathematics lessons across the school system. The dimensions of the Mathematics Framework are intertwined: they are not separate. Whilst encouraging learners to use their own strategies, you should also assist them to develop efficient and accurate procedures that build on conceptual understanding and draw on their reasoning skills.

To give teachers a clearer idea of how this could happen we now provide some more examples of how the four dimensions might be visible in mathematics classrooms. These examples are given for each of the four phases in the schooling system. The exemplars are not prescriptive nor are they exhaustive. Visual representations that are shown can vary. These are also not prescriptive and teachers need to decide which representations would be most effective in their particular contexts.

The exemplars include examples of learners' working (numeric and symbolic) from a range of different topics from the mathematics curriculum. The examples were chosen to address different mathematical topics from the curriculum but because of the nature of this document, they do not (and cannot) cover the entire curriculum. They serve as exemplars of the application of the framework in the context of mathematical content specific to each phase. Many of the exemplars include other representations as well, such as diagrams, sketches or concrete demonstrations. The exemplars are all discussed to highlight the way in which the dimensions of the framework play out in classrooms. Considerations for teaching and benefits for learning related to the given examples in the context of the dimensions they illustrate are also discussed.

The exemplars are written up in slightly different ways across the different phases. All of them highlight meaningful engagement with curriculum ideas on the part of teachers and learners. The emphasis on teacher and learner engagement varies. This variation is purposeful as variation is key to meaningful curriculum implementation. Teachers are encouraged to study the exemplars from their own phase but also to read and learn from the exemplars presented for other phases, particularly the phases linked directly to their own phase to support smooth transitioning between phases.

3.4.1 Foundation Phase

There is a growing interest in including mathematics among the learning goals for young children and improving the teaching of mathematics in developmentally appropriate ways. If young children are not enabled to develop appropriate knowledge, skills and understanding of mathematics in the early grades, the achievement gap is entrenched from the earliest years, extending the intergenerational cycle of lost opportunity and wasted potential. The foundational knowledge and skills for mathematical learning in later years is laid in this phase – beginning in Grade R and continuing into Grades 1-3. The richness of all of the work covered in these four grades cannot be fully represented here hence teachers will need to apply the learning from these exemplars across the curriculum.

(a) Conceptual understanding

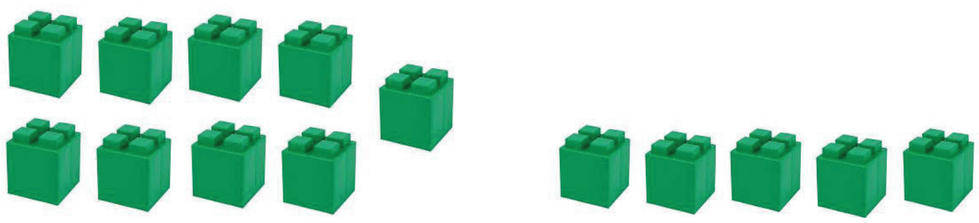
The following example demonstrates how when learners work with numbers in a meaningful way they are given the opportunity to develop their conceptual understanding. This conceptual understanding grows while learners reason about the number work they are doing, sometimes by using different strategies.

> The teacher writes the sum on the board:
>
> 9 + 5 = __
>
> Then the teacher asks the learners: *Is the answer greater than 10? How do you know?*
>
> The learners are asked to do the following: *Without calculating the answer - discuss how you know this with a partner and argue for why this must be the case.*
>
> The teacher explains: *You can use counters (or other aids) or write out your explanation with numbers to explain your reasoning.*

When learners find ways to answer the question '*Is the answer greater than 10? And how do you know?*' without simply calculating the answer they learn to reason in many different ways. The reasoning component of the discussion, and reasoning correctly, is more important than simply 'getting the answer correct' since it supports conceptual development. It also supports developing learners' logical reasoning rather than simply their ability to calculate accurately since the framework dimensions are linked – they do not work in isolation.

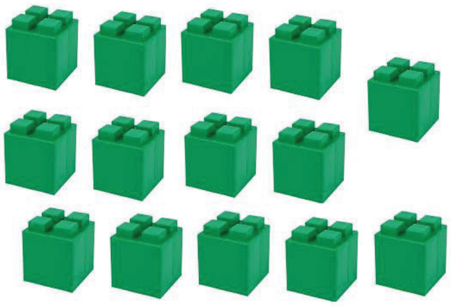
There are key representations that teachers and learners can use in order to develop central mathematical concepts. A range of representations are given below, followed by a discussion comparing the differences between them and their relative usefulness in the development of number sense, number concept and operation concept.

Learner 1

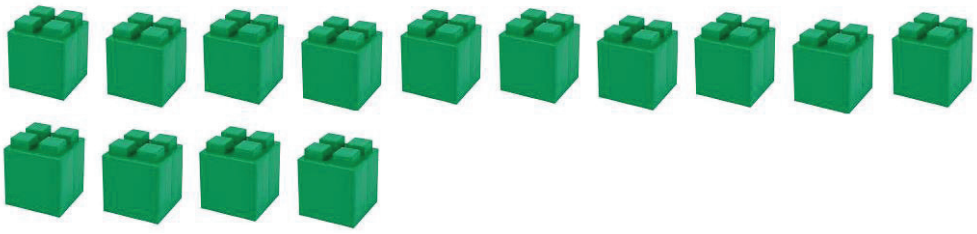


Learners could use counters as follows: I lay out 9 counters and 5 counters. I think 9 + 5 is more than 10.

I combine all of the counters:



I regroup and count the counters to find my answer. I find that 9 + 5 is greater than 10 because I have one group of 10 and 4 extra counters. So 9 + 5 = 14

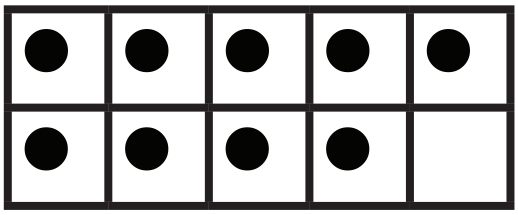
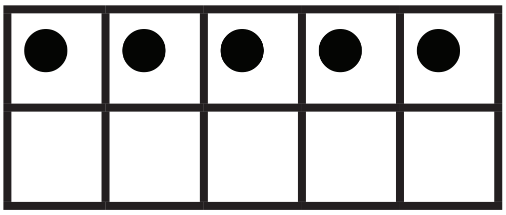


Learners do not need to layout the 4 more – they could say I only need one from this group to get (or make) ten here and so I have some left over so I can see its definitely more than ten.

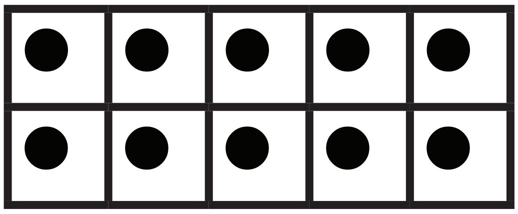
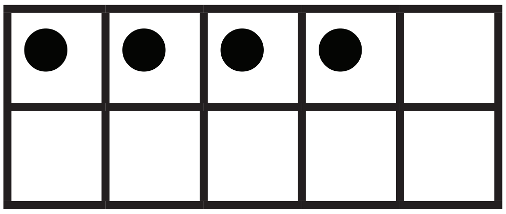
Learner 2

Learners could use 10 frames to do the same thing as follows:

They could show 9 on the first 10-frame and 5 on another

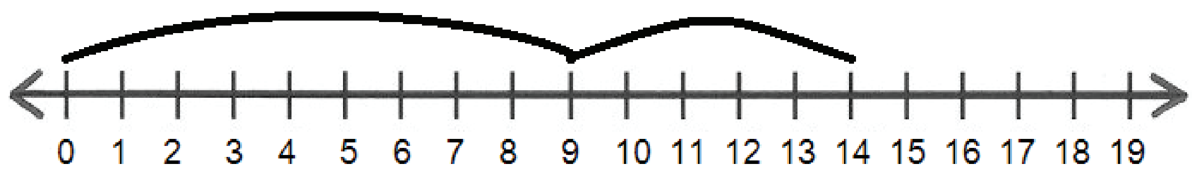
The learner could then complete the first 10-frame with one from the 5 which would leave 4 on the other 10-frame

So we see we have 10 and 4 and we can see that the answer is greater than 10 because it fills more than one 10 frame. In such questions learners do not need to focus on the answer to the calculation 9 + 5 but rather on thinking logically about how big the answer will be compared to 10 and on developing mathematics language to explain their thinking.

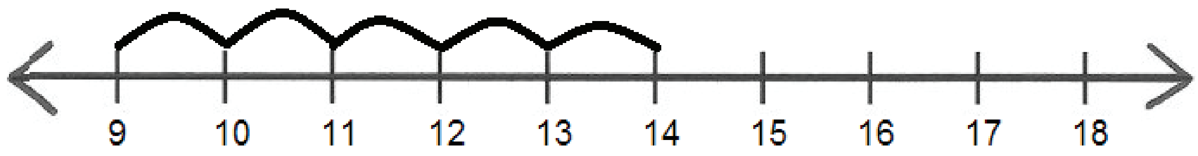
Learner 3

Learners could also use the number line to explain their reasoning. They could do so in different ways. The number line reasoning below shows a high level of abstraction, since the learner has shown using two jumps that shows that 9 + 5 equals 14, which is greater than 10.



Learner 4

Learners could use a number line, choosing where to place the starting point of 9 and then counting on. The number line reasoning below shows a leaner counting on in 1s from the number 9. Counting on from 9 shows a certain level of abstraction, although counting on in 1s is a low-level strategy. The strategy used does show how the learner has proved that 9 + 5 = 14 which is greater than 10.



Learner 5

At a higher level of abstraction, learners could show their reasoning numerically or argue numerically such as:

9 + 5 = 9 + 1 + 4 = 10 + 4 (which is bigger than 10).

A learner could also say: ‘since 9 + 1 is 10, if I add more than one the answer has to be bigger than 10’.

CONSIDERATIONS FOR TEACHING

The concept of addition is introduced using small numbers but it is developed as the number range increases. In this example the first question the teacher asks is, ‘*Is the answer greater than 10?*’ At this point, the teacher should allow learners to answer the question with a ‘yes’ or a ‘no’. This commitment will maintain the learners’ interest in the discussion and activity that follows when learners investigate ways of proving their explanation to the second question, ‘*How do you know?*’ using counters, ten frames, or other means. (Teachers should discourage learners from simply giving the answer ‘14’ because the reasoning is the focus of the discussion, not the answer.) Giving explanations is a meaningful mathematical activity for young learners. The explanations are essential to the activity because through them learners build their conceptual understanding. Learners must be given opportunities to speak mathematically – to use mathematical language – to express their mathematical ideas based on mathematical work they have done.

In the Foundation Phase the concept of addition is taught, first with examples where the answer is less than 10 and then with examples where the answer bridges 10. An example of addition where the answer goes over 10 (bridges the 10) draws on learners’ conceptual understanding of number (place value) and addition (combine all to find the total). Learners need to be able to explain why the answer to 9 + 5 is greater than ten.

Learning benefits associated with teaching for conceptual understanding of addition bridging 10 in Grade 1

If teachers allow learners to use concrete or visual aids to help them develop their mathematical reasoning and subsequent explanations to questions such as the one above this will support strong conceptual understanding of the relationship between numbers and to develop number sense. This lays a solid foundation for operational activities in the higher grades. A ruler could be used as a concrete representation of a number line.

The five strategies above show different ways of learners establishing for themselves that the answer to 9 + 5 is greater than 10. The first one is less structured. The second one builds on the structure of the 10 frame which allows learners to see more clearly when the 'ten' has been completed. The third and fourth shows the reasoning using a number line and the fifth learners shows reasoning by using the numeric symbolic representation.

Teachers should encourage and engage with all strategies given and explained by learners. The structure of the 10 frame allows learners to see that 1 more onto 9 completes the 10 structurally. This could lead more effectively to the conceptual understanding of number and grouping into tens. We want learners to move on from counting and calculating to seeing and using the structure of numbers and relationships between them.

It is precisely this understanding of relationships that enabled young Gauss to, at his teacher's request, add all the numbers from 1 to 100 i.e. 1 + 2 + 3 + … + 97 + 98 + 99 + 100 almost instantly – he used his knowledge that the order of addition doesn't matter and paired up numbers to make 100s (i.e. 1 + 99; 2 + 98; 3 + 97, …). He got 49 pairs of numbers making 100. This together with the remaining 50 in the middle and 100 at the end, makes 4900 + 50 + 100 = 5050).

(b) Procedural fluency

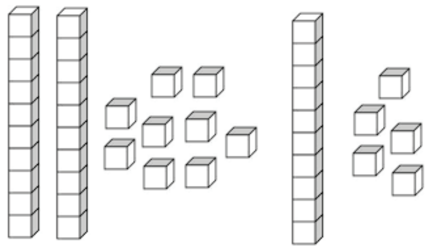
The curriculum suggests several strategies for learning in the Foundation Phase. Many of these are shown below. Remember that the framework dimension are linked – so developing learners' procedural fluency in performing calculation strategies will enable them to build up their mathematical learning according to the other framework dimensions.

29 + 15 = __   Find the sum using any method.

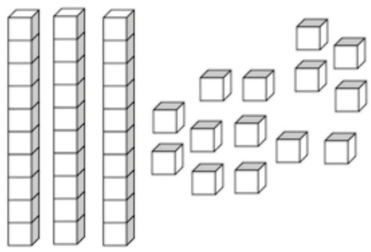
Learners could add these two numbers using a variety of strategies using different mathematical procedures that build on the conceptual understanding of addition.

For example: Using base ten blocks a learner could build the displays of the two numbers and then put all the blocks together to find the sum. They should be encouraged to group blocks in tens so that when they put them together it is easy to count the groups as 'ten, twenty, thirty, forty, …'. This concrete method is useful if learners have not yet grasped the place value concept of exchange when the two numbers that are being added in the units place bridge 10.

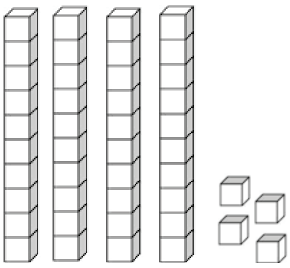
First, lay out the two displays:



Second, combine the tens and the units:



Finally work with the units to regroup – this working build on the work done in the example above when units are added, and the answer is bigger than 10. The working with the units blocks give me 1 ten and 4 extra units.

The final answer is 44: 

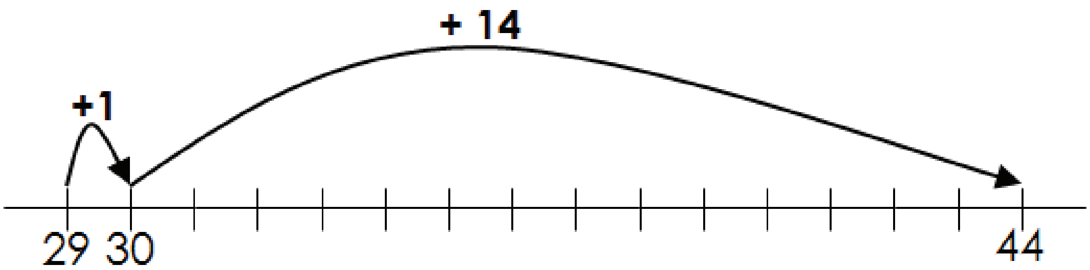
If a learner is already comfortable with addition of two single digit numbers that bridge ten, numerical procedures can be used, such as:

29 + 15 = 20 + 10 + 9 + 5 = 30 + 14 = 44

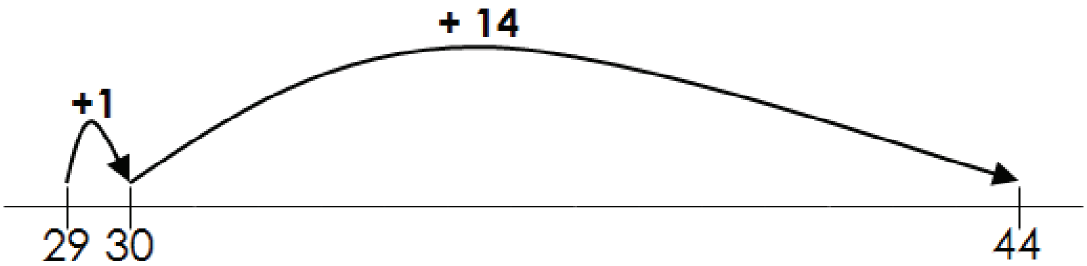
Other numerical methods or number lines could also be used.

29 + 15 = 29 + 1 + 14 = 30 + 14 = 44

This could be shown on a number line as follows:

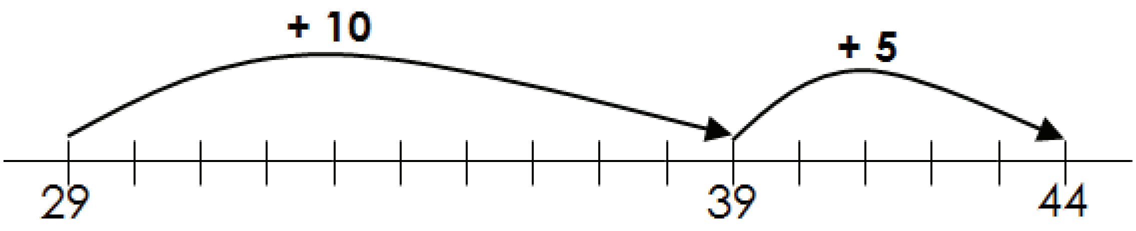


It could also be shown on an open number line:



29 + 15 = 39 + 5 = 44

This could be shown on a number line:



It could also be shown on an open number line:

Considerations for teaching procedural fluency

Procedural fluency is developed though practice but it has to be built on a strong conceptual foundation. To teach for procedural fluency, teachers need to give many examples and should vary these examples.

Much of the value of developing quick recall of basic bonds in the Foundation Phase is that it develops the ability of learners to use rapid recall when necessary, to carry out strategic calculating, when working with bigger numbers and to develop their ability to think strategically when doing mathematical calculations.

Learning benefits associated with developing procedural fluency

In the Foundation Phase there are several fluencies in number that must be developed and are part of the curriculum (appearing often in mental maths sections). These fluencies enable learners to work confidently and competently with numbers.

For example:

- Bonds to 10 for example, 7 + _ = 10 which can be extended in higher number ranges for example, 27 + _ = 30
- Multiples of 10 for example 70 + _ = 80
- Doubling: double 6 is 12; double 35 is 70; double what is 42?

(c) Strategic competence

We provide two examples to illustrate learners' own strategies in the Foundation Phase.

Example 1 provides a series of different strategies using additive operations. Example 2 involves an open-ended question about routes on a grid. Once again teachers are reminded that the dimensions are linked – developing learners' strategic competence will enable development of the other dimensions of their learning of mathematics.

Note that it is unlikely that learners will be able to use a range of strategies if they have only ever seen one strategy. Thus teachers should point all learners' attention to a range of effective strategies. Teachers should also nudge learners towards the use of increasingly efficient strategies over time.

Example 1: A series of different strategies for additive operations

Strategy 1: Decomposition of one number to make 10

A. 34 + 7 =

One learner could move from counting on to bridging through ten e.g.

34 + 7 = 34 + 6 + 1 = 40 + 1 = 41.

This strategy:

- builds on the fluency of knowing bonds to ten;
- identifying the nearest 10.

Similarly, another learner could apply the same strategy but decompose the first number since 7 is nearer to 10 e.g.

30 + 1 + 3 + 7 = 31 + 10 = 41

B. 526 + 38 =

A learner could do the following calculation:

526 + 38 = 526 + 4 + 4 + 30 so 530 + 34 = 564

This strategy

- builds on the fluencies of knowing bonds to 10;
- knowing how to add ten and multiples of ten to any number;
- knowing how to add a single digit to a multiple of 10 (e.g. 60 + 4 = 64).

Strategy 2: Decompose numbers into place value

A. 23 + 36 =

23 + 36 = 20+ 30 + 3+ 6= 50 + 9 = 59

B. 18 + 23 =

18 + 23 = 10 + 20 + 8 + 3 = 30 + 11 = 30 + 10 + 1 = 41

Each addend is decomposed (or broken down) using place value so that tens and ones can be combined separately. This makes the calculation easier to do. This strategy begins to develop in grades 2 and 3 and is based on an understanding of place value since it groups the units, tens and hundreds to be added. In **B**, a learner has worked similarly to **A** but indicates how the units are grouped into tens

Strategy 3: Pairs of numbers that make 10

26 + 12 + 4 + 18 =

20 + 6 + 12 + 4 + 10 + 8 =.

It is easy to use mental computation if you look for pairs of numbers that together make 10 or multiples of 10. To do this, learners could pair numbers that make 10 or multiples of 10.

Strategy 4: Use a doubling strategy and then compensate for the difference

13 + 14 =

Double 13 and add 1 (26 + 1 = 27)

or double 14 and subtract 1 (28 – 1 = 27)

This technique requires a strong number sense and knowing one's doubles fluently. Learners who are able to choose such techniques are quite flexible in the strategies they use.

Strategy 5: Compensation

Teachers may encourage learners to use compensation as a strategy to develop their mathematical skills. Compensation (in mathematics) is a process of reformulating an addition, subtraction, multiplication, or division problem so that it can be calculated more easily.

The goal of compensation is to manipulate the numbers into easier, friendlier numbers to add, subtract, multiply or divide. The difference (in compensation) from other strategies is that when you compensate a 'paired' change is made: **one action is compensated by another action.** When you compensate you need to remember what change you have made, and adjust for this change, so that you do not change the question itself.

The examples shown below involve addition and subtraction, where compensation normally involves changing one of the numbers into a multiple of 10 which is easier to work with.

In addition, if you subtract a specific amount from one number then you must add that amount to the other number.

A. 78 + 27 =

It is simpler to add to a multiple of 10. We can change 78 to the nearest multiple of 10 by adding 2:

We know that 78 + 2 = 80.

But if we add 2 to the one addend, we need to compensate for this by subtracting 2 from the other addend. 27 - 2 = 25.

Thus we can change the sum to 80 + 25 = __ (and the solution can easily be seen to be 105 because the compensation has simplified the calculation.)

In subtraction, if you add a specific amount to one number then you must add that amount to the other number.

B. 78 – 27 =

It is simpler to subtract a number that is a multiple of 10.

We can change 27 to the nearest multiple of 10 by adding 3

BUT then we need to compensate for this by then adding 3 to the number we are subtracting from as well:

So we change the subtraction to (78 + 3) – (27 + 3) = 81 – 30 = 51

Considerations for teaching

Learners in the FP should be discouraged from using unit tallies or counting when doing operations except in the initial stages of calculating and even then they should be encouraged to structure their tally lines in rows of ten or groups of ten rather than randomly. Unit counts (little drawings of circles, lines or dashes for examples) should not be accepted as records of a calculation – numeric working should be used to express the calculation once it has been completed. Learners should use efficient strategies such as bridging through 10 and they should learn their bonds to help them do this work mentally.

Focusing on strategies is important for developing increased efficiency. This builds procedural fluency (noted above) which needs to be developed.

In the Foundation Phase, teachers can also consider the column method for recording numeric working. The column method is an efficient recording strategy that uses the structure of numbers in our base ten numeration system. It must be properly explained using place value in order for learners to make good use of it. Teachers should always encourage learners to check their solutions using a variety of methods.

The use of columns to record addition has not featured explicitly in the Foundation Phase CAPS although many young learners are shown this method by their siblings, parents and teachers. Columns can be used very efficiently to structure numerical calculations because they formally represent place value.

Place value concept is established in the Foundation Phase – in Grade 2 learners are introduced to 2-digit numbers (which are numbers that have digits in two places – tens and units) and in Grade 3 they are introduced to 3-digit numbers (which are numbers that have digits in three places – hundreds, tens and units). The use of columns to record numerical calculations, when the method is explained using conceptual links, consolidates conceptual understanding while facilitating efficiency of calculation. This strategy is thus included in the examples presented below. Teachers are reminded of the integral nature of the four dimensions of the framework – the link between conceptual understanding and procedural fluency should be remembered at all times.

For example: Learners can use columns to work meaningfully and efficiently with 2-digit numbers if they talk about what they are doing using place value.

e.g. 34 + 27 = __

T	U
+1	
3	4
2	7
6	**1**

To add 2-digit numbers you line them up using columns for tens and units. The units must be in the same column and the tens must be in the same column.

Start by adding the units. 4 + 7 = 11
11 is one ten and 1 unit. I must break this into tens and units.

I must carry the 1 ten into the tens column, so that I can add it to the other tens. The single unit remains as part of the answer in the units column.

I now add in the tens column. I have 3 tens plus 2 tens which is 5 tens plus one more ten which I carried over from the units column. That gives me 6 tens altogether. I write it in the tens place of the answer.

The answer to 34 + 27 is 61.

The number work of exchanging the ten units for 1 ten should be linked to concrete work done using base ten blocks where learners see how to exchange ten units for 1 ten. This will guide them to understand the conceptual structure embodied in the algorithm. This conceptual work underpinning the procedure (which again highlights the interconnectedness of the framework dimensions) will prevent learners from following the steps of the procedure without thinking about the meaning behind what they are doing. This applies to all methods of calculation.

As learners progress the column method can be extended when learners begin to work with 3-digit numbers and do calculations with bigger numbers. Learning how to use this method of recording calculations with 2- and 3-digit numbers lays the foundation for generalisation of the method to even bigger numbers in the Intermediate Phase. If learners do not start using this efficient and structured recording strategy early it will be difficult for them to make sense of it when they do calculations with bigger numbers in the Intermediate Phase and in later years.

e.g. 526 +38 =

H	T	U
	+1	
5	2	6
	3	8
5	**6**	**4**

The teacher should explain all of the steps in the working shown in the written calculation to the learners. All grouping and exchanges carried out while doing this kind of calculation should be demonstrated by doing concrete work with base ten blocks and making reference to the place values involved in the exchanges, so that learners will understand conceptually how the exchanges between columns is done.

Learning benefits associated with using a variety of own strategies

Learners who are encouraged to use a variety of strategies develop their competence in and knowledge of a range of strategies, which enables them to work more confidently when doing mathematical operations.

Learners should be exposed to, and be allowed to try out, many different strategies for doing calculations. This will enable them to judge for themselves which is the most efficient strategy and (or) which strategy they prefer to use. Numeric strategies must be emphasised and encouraged. Efficiency of numeric recording should also be encouraged. Learners should draw on mental mathematical knowledge in order to do calculations quickly and efficiently. Formal numeric records of calculations should accompany all activities that are designed to develop mathematical number sense in the context of number and operations.

The long term benefit of using strategies that uses the structure of place value when recording numeric working from the start is that these strategies can more easily be generalised to bigger numbers in later years. Counting as an operation strategy should be actively discouraged, especially when they are inefficient such as when working with large numbers. This will help to move learners away from using tallies (which represent counts) when they operate on numbers. The column method of recording numerical working is an efficient recording strategy. Alternative strategies are useful in addition to the column method because they highlight other aspects of number that also build learners number sense. Relationships between different methods of recording numeric calculations should be discussed as this kind of discussion supports the development of conceptual understanding. Such discussions also promote the development of learners' reasoning ability which is the fourth dimension of the framework.

Learners will develop the ability to work flexibly with numbers and perform operations efficiently if they are given many varied opportunities to develop their procedural fluency at the same time as they build up their conceptual understanding. The connection between conceptual understanding and procedural fluency could be deepened in the process of learners using and refining their own strategies.

The next example is one where learners need to think of 'out of the box' in order to find the solution. This is an example of the third type of strategy for this framework dimension.

Example 2: Look at this grid of numbers:

7	11	15♠	19
23	27	31	35
39	43	47	51
55	59	63	67

A spade icon is placed on a block with the number 15. Roy moved the spade icon to block number 59. He only moved straight across and straight up/down. He did not cross over a block more than once.

- Give as many possible routes that he may have taken to move the icon.
- Which route is the shortest?
- Which route is the longest?
- Describe your routes. How do you know you chose the longest/shortest route?

Solution: There are several routes – learners could explain many of them.

Shortest route – 4 blocks. There are two possible shortest routes:

- across 1 and down 3 or
- down 3 and across 1.

Longest route – 14 blocks: There are two possible longest routes:

- right 1, down 3, left 1, up 2, left 1, up 1, left 1, down 3, right 1 or
- right 1, down 3, left 1, up 2, left 1, up 1, left 1, down 2, right 1, down 1.

Considerations for teaching

If the teacher asks all learners to give their routes the class could discuss the answers to the questions about the shortest and longest routes by comparing their answers. They could also work collectively to work out the maximum number of routes they can identify on the table. Teachers need to make sure that learners know how to count a route – that a move involves going from one block to the next and that you cannot re-count the block that you landed on.

Teachers could discuss the way in which the two longest and shortest routes are related – the one is the reverse of the other. The shortest route can be argued as such – there is no shorter way to move between the two given blocks, if one has to count a route as instructed. The longest route covers all of the squares between the two given blocks and so it must be the longest block.

Learning benefits associated with 'out of the box' thinking

When learners have to think about and explain their answers it develops their ability to speak mathematically and reason using mathematical ideas. This builds their confidence and deepens their knowledge. It prepares them for mathematical activities that involve reasoning.

(d) Reasoning

The fourth dimension example focuses on the reasoning dimension. You will notice that here learners are required to find solutions without doing calculations. They do this using information which is given – in this case, a statement about a sum of two numbers. In order to reason, learners draw on (and at the same time build up), their conceptual understanding.

Given that: 426 + 38 = 464, what is

- 426 + 39 =
- 464 – 38 =
- 38 + 426 =
- 424 + 40 =

The questions above ask learners to solve a group of questions by relating the questions to information that has been given i.e. that 426 + 38 is 464.

They must use **reasoning** that relates to this given fact to find the answers. They use deductive reasoning to find the answers. The questions could be reasoned out as follows:

Given that: 426 + 38 = 464, what is:

- 426 + 39 = 465

To work out this solution learners reason by saying 'if I add one more to one of the numbers being added then I should add one more to the answer' (thus the answer becomes 464 + 1= 465)

- 464 – 38 = 426

To work out this solution learners reason using their knowledge of the relationship between addition and subtraction – that these are inverse operations. (thus the solution is 426)

- 38 + 426 = 464

To work out this solution learners reason using their knowledge that addition is commutative. (If 426 + 38 = 464 then 38 + 426 = 464)

- 424 + 40 = 464

To work out this solution learners could draw on their knowledge of compensation. The learner identifies that in this question, the addends have been changed in a way that the overall sum has not been changed. The number 2 has been subtracted from 426 and then added to 38, changing the written sum to 424 + 40. Hence the solution is the same, 464.

Considerations for teaching

To find the solutions in this activity learners must reason based on what they have been given rather than calculate. They are not meant to do the numerical calculations. At the level of the Foundation Phase and in the number range for this phase it is possible to challenge learners with appropriate questions such as this one using reasoning about operations. These activities call on their adaptive reasoning ability. The way in which the question is set - giving certain information and asking 'what is' calls on the learners to reason (rather than calculate) and then need to be able to justify their answers.

Learning benefits associated with the ability to reason using relationships between numbers

The reasoning shown above is often referred to as adaptive reasoning – it involves using knowledge of the number relationships to solve problems rather than using calculating strategies. As shown in the examples, learners use reasoning about additive operations to find the solutions. Doing such activities helps learners to develop the capacity to think logically and be able to analyse the relationships among concepts. They learn how to reason in order to justify why a procedure has worked.

These activities will develop independent thinking skills and build the learners' confidence in mathematics.

3.4.2 Intermediate Phase

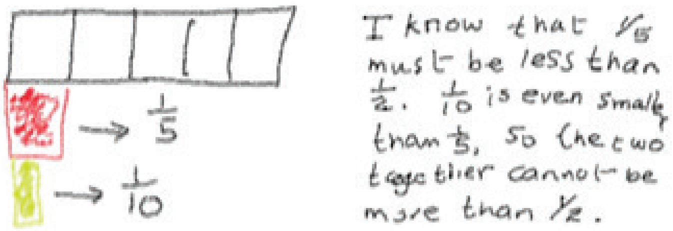
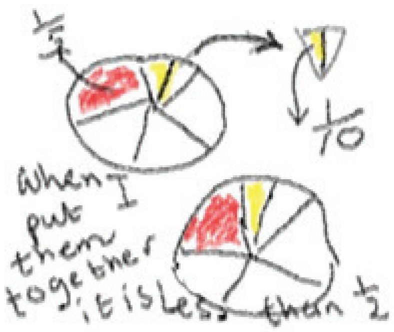
The four dimensions of this framework are all intertwined, none of them stand alone. Each topic and piece of work will involve aspects of all four dimensions.

(a) Conceptual understanding

Learners cannot develop a sound conceptual understanding of a topic or aspect of mathematics without applying mathematical reasoning. Where unstructured problems like the one below is used, learners will use and discuss a range of strategies which will in turn develop their strategic competence.

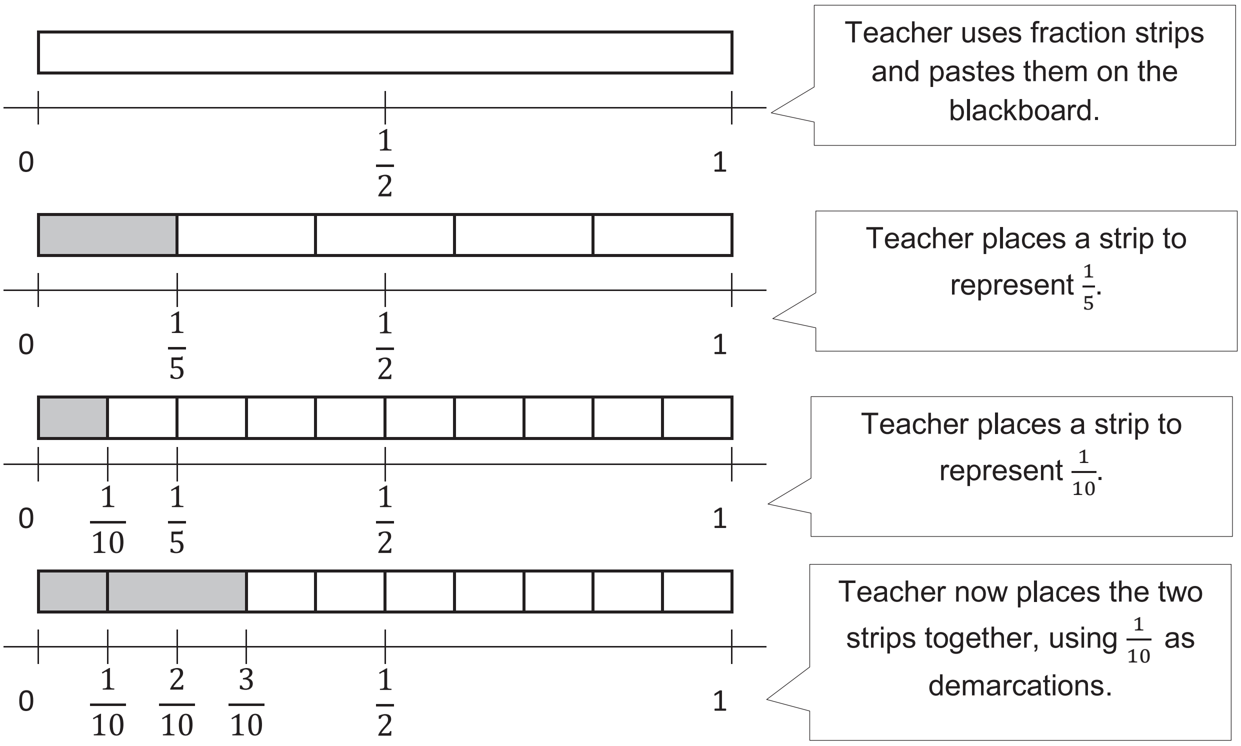
The teacher poses the following question to her learners: Think about the following: $\frac{1}{5} + \frac{1}{10} = \_\_\_$ Is the answer more or less than $\frac{1}{2}$? How do you know?

At this point, learners have not yet been taught the algorithm for the addition of fractions. Below are attempts of two learners.

Ntombi's response	Juno's response
	

The way in which the question was posed, opens the door for the conceptual development of the size of fraction parts. In learners' responses they reasoned about fractions as being a 'part of a whole'.

To move from the diagrammatic representation of fractions as parts of wholes and to develop the idea that a fraction is a number (having a position on the number line), the next step could be to use the representation that Ntombi made (the rectangle). The rectangular partitioning lends itself to fitting with demarcations on a number line. The 'whole' now represents 1 unit on the number line.



What the teacher did was to transform the representation of 'fractions as parts of a whole' into the notion of fractions as numbers, by using a number line.

When writing on the board: $\frac{1}{5} + \frac{1}{10} =$, she probes learners to see that $\frac{1}{5}$ is the same as $\frac{2}{10}$

(show that equals)

She demarcated the number line into tenths in order to show that .

By putting the two parts next to each other on the number line, the teacher shows the learners that the answer is $\frac{3}{10}$ and it is also clear on the number line that $\frac{3}{10}$ is less than $\frac{1}{2}$.

So we have $\frac{1}{5} + \frac{1}{10} = \frac{2}{10} + \frac{1}{10} = \frac{3}{10}$ which is less than .

We can say that two-**tenths** plus one-**tenth** is equal to three-**tenths,** which is less than one half. The teacher should speak about the addition using the fraction word names, since this emphasises the conceptual basis for the addition of tenths.

Procedurally, this can be simplified to 'adding the numerators of like fractions' once the conceptual basis for this addition has been established.

Note that the teacher did not start with the algorithm. Discussions around the role of the numerator and denominator should follow conceptual demonstrations and discussions to enable learners to understand the idea that we can add fractions of the 'same kind'.

Considerations for teaching

Learners often find it difficult to understand that a fraction is a number. In the example above, when the two fractions are positioned on the number line, this is a powerful demonstration of a fraction as a 'part of a whole' and a fraction as a number with its own position on the number line. This demonstration can be used to help bridge the gap between understanding fractions as parts of wholes and fractions as numbers. It can also be used to demonstrate the addition of fractions.

When a new concept is introduced to learners, the teacher should plan a lesson for conceptual development. By using unstructured problems like the example above, teachers allow learners to generate a variety of strategies to find solutions. It is important for learners to discuss their viewpoints with their peers. Learners should be encouraged to use words to express their understanding. The teacher should use these ideas to challenge learners to construct and connect ideas through reflective discussions. The more the learners are exposed to this type of teaching, the more likely it is that they will develop into problem solvers. Lastly is should be emphasised that a learner with conceptual understanding is more able to transfer this knowledge to new situations and apply it to new contexts.

Learning benefits associated with conceptual understanding

Both the learners' diagrams and the teacher's illustrations of $\frac{1}{5} + \frac{1}{10}$ help learners to understand the relative size of fractions. Instead of just illustrating $\frac{1}{5}$ or $\frac{1}{10}$ as separate fraction parts of different wholes, this example guides the learner to develop a more holistic approach to the interpretation of the relative size of fraction parts.

Teachers often prefer to use *halves* and *quarters* when they do demonstrations, but in the Intermediate Phase, learners should be able to work with other fractions as well since fraction concept needs to be generalised to all fractions – with different denominators and numerators. This example has worked with unit fractions of fifths and tenths. Teachers in the Intermediate Phase should do practical demonstrations with other pairs of fractions such as $\frac{1}{3}$ and $\frac{1}{6}$, $\frac{1}{6}$ and $\frac{2}{9}$, $\frac{3}{8}$ and $\frac{5}{12}$, , and so on, varying the numerators and the denominators.

If conceptual understanding is gained, then learners should be able to build on this conceptual understanding to reconstruct procedures, if they have been forgotten. (If procedural knowledge is the limit of a learner's learning, they may struggle to remember these procedures over time.) In the example above learners will have seen that they need to change fifths and tenths into the same kind of fraction (tenths) in order to add them. This demonstration builds on and consolidates the learners' understanding of equivalence by showing a way in which equivalent fractions can be used. Connecting the concepts of equivalence and addition strengthens the broader understanding of the learner.

(b) Procedural fluency

Fluency is developed through repeated practice. This practice will build the conceptual understanding that is the basis of any calculation. Learners need to do as much work as possible in order to become fluent – you should give them as many examples as possible to help them develop this essential mathematical skill. Procedural fluency allows learners to free their minds to reason mathematically and to experiment with various strategies.

Calculate

$$\frac{1}{5} + \frac{1}{10} =$$

Learners could show their working for this sum in different ways one example is:

$\frac{1}{5} + \frac{1}{10}$

$= \frac{2}{10} + \frac{1}{10}$ (Procedure Step 1: use known equivalent fractions.)

$= \frac{2\ +\ 1}{10}$ (Procedure Step 2: add fractions with the same denominator.)

$= \frac{3}{10}$

Considerations for teaching

The CAPS for the Intermediate Phase specifies that learners should be able to add and subtract fractions with like and unlike denominators. If the algorithm for addition of fractions is developed without the conceptual background (in this case it is the knowledge that unlike fractions cannot be added and that equivalent fractions need to be found in order to add), learners will just follow procedures that they do not understand. When another algorithm (for example for multiplication of fractions, which works differently) is encountered (when learners move into the Senior Phase), these two different yet related fraction operation algorithms could become jumbled up and learners could get confused.

Often teachers use terms like ‘2 over 10’ to say $\frac{2}{10}$. This can lead to the misconception that each fraction consists of two different whole numbers (one above the line and one below the line).

The following incorrect procedure is an example of this misconception, which is often used by learners.

$$\frac{1}{5} + \frac{1}{10} = \frac{1\ +\ 1}{5\ +\ 10} = \frac{2}{15}$$

***Misconception*: a misunderstanding -** that you add the numerators and add the denominators when you add fractions.

Once learners have learned to multiply fractions (in the Senior Phase) using the procedure ‘multiply the numerators and multiply the denominators’, they may incorrectly apply this procedure to addition of fractions – add the numerators and add the denominators. This is an example of procedures getting confused.

In the next exemplar we discuss strategic competence. Whilst working with learners’ own strategies is important, it is also important that learners develop generalisable procedures (guided by the teacher) that will allow them easily and efficiently to arrive at the correct answer. The unfortunate scenario in most schools is that procedural knowledge takes precedence over conceptual understanding. Teachers emphasise the ‘what’ or the ‘how’ instead of the ‘why’.

In teaching about fractions the phrase ‘what you do to the top you do to the bottom’ is an example of a focus on ‘what’ or the ‘how’ instead of the ‘why’ which leads to misconceptions such as the one shown above where adding is done by ‘doing the same thing to the top and the bottom’. The ‘doing’ is not specific enough and ‘why’ it works is not contained in the phrase, which over-simplifies the process of finding equivalent fractions. Teachers should always give full and proper explanations when they teach about mathematical procedures so that the procedures can be properly generalised and established in the minds of learners.

Learning benefits associated with procedural understanding

Conceptual understanding and procedural fluency, are not opposing processes, they should not be seen as in competition to one another. Conceptual understanding and procedural skills are equally important, and both are assessable using mathematical tasks of sufficient richness.

Learners who have developed an understanding of a concept, and know the relationship between the concept and the procedure, will be able to apply the procedure. For example when adding fractions learners will eventually know that like fractions must be added, and will convert fractions if they need to, without necessarily thinking, 'I am using the concept of equivalence to change the denominators of the fractions so that I will be able to add them'. Procedures need to be understood (based on conceptual understanding) and then used often for learners to become comfortable in using them.

The more learners use procedures to solve problems, the more skilled they will become. They will develop the necessary fluency in performing the procedures. This procedural fluency will build up learners' confidence and enable them to think mathematically.

(c) Strategic competence

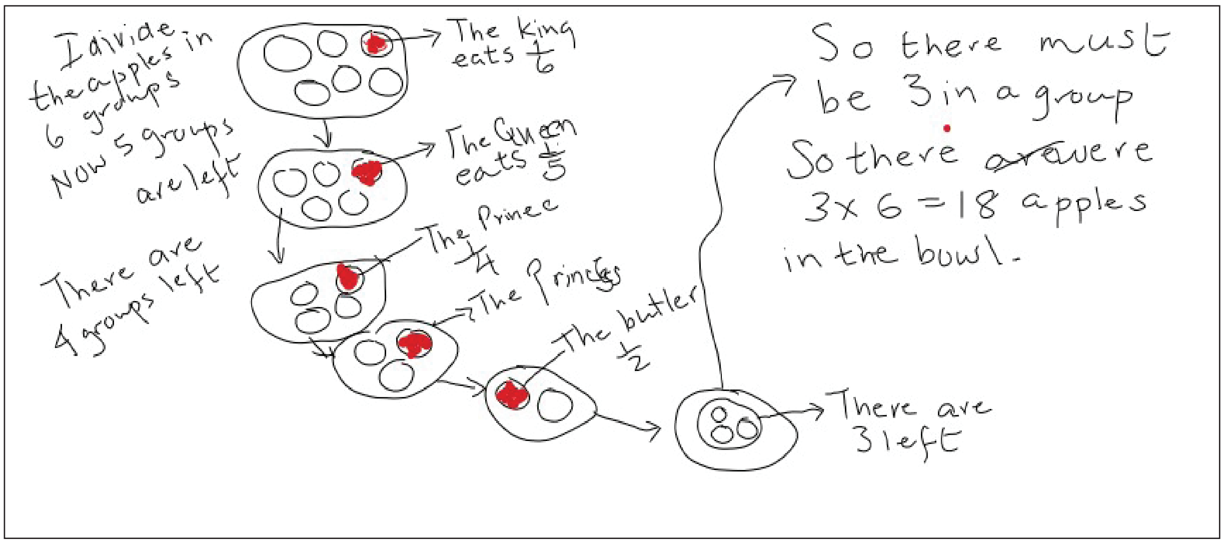
As mentioned before the four dimensions of this framework are all intertwined, none of them stands alone. Each topic and piece of work will involve aspects of all four dimensions. Similarly, the examples of strategic competence below, build on learners' conceptual understanding and draw on their procedural fluency. They also involve the important dimension of reasoning.

We provide two examples to illustrate learners' own strategies in the Intermediate Phase. Example 1 is about fractions of whole numbers. Example 2 focuses on division.

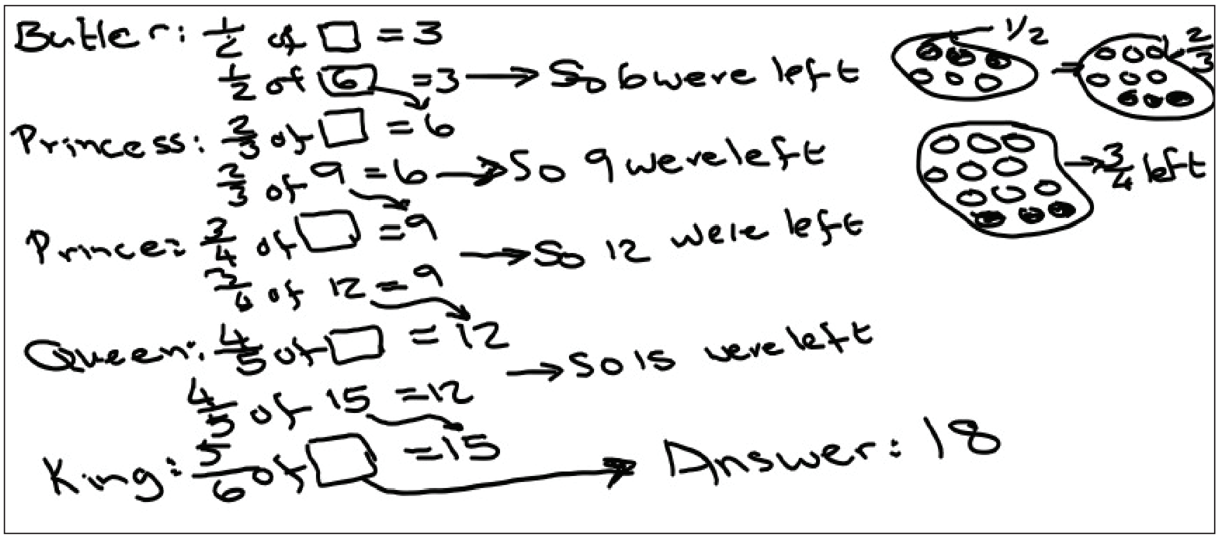
Example 1:

There was a bowl of apples on the dining room table in the palace. At midnight, the King felt hungry, went to the dining room, and ate $\frac{1}{6}$ of the apples. After a while, the Queen got up and ate $\frac{1}{5}$ of the apples left over. Then the Prince got hungry, went down and ate $\frac{1}{4}$ of the apples left in the bowl. The Princess was next and ate $\frac{1}{3}$ of the apples left. The butler saw all this and ate $\frac{1}{2}$ of what was left. There were 3 apples left in the bowl. How many apples were in the bowl before everybody started to eat the apples?

Learner 1: Bongi used a pictorial approach. He started with King's portion: He made 6 groups (represented by the small circles) of which the King ate one-sixth. He was left with 5 groups, of which one fifth was eaten, and so on. He worked his way down to the butler who ate a half of two groups, leaving one group on the table. Bongi then connected the number of apples left on the table to the size of the group and used this to work out how many apples there must have been to start with.



Learner 2: Lerato started with the butler's portion, and worked her way up until she got to the King's portion. Lerato used the fraction part that was known each time (the remainder) to work out what the whole should be. Starting with the Butler, she knew that the Butler ate a half and got 3 apples, so he must have eaten a half of 6. She built this all the way up to what the King ate.



Considerations for teaching

In the example above Bongi and Lerato used different entry points to solve the problem. Many problems can be solved in several different ways, using different strategies. Teachers should allow learners to use their own strategies, and not assume that the strategy the teacher has presented is the only one, or even the best one.

Teachers need to create a classroom culture that encourages learners to talk about their understandings of the problem and their solution strategies. Learners may struggle to express their thinking. Teachers should support learners to develop appropriate mathematical language (both verbal and symbolic).

Learning benefits associated with learners being able to use their own strategies

In the example above, learners find fractional parts of a collection of apples. As learners work through the example above, it will consolidate and further develop their understanding of fraction parts of collections of objects (set model). Learners may not develop the same understanding if they are only expected to follow a standard procedure to find numerical answers to a list of fractions of wholes. Simply following a procedure repeatedly seldom develops sound conceptual understanding.

Encouraging learners to use their own strategies empowers learners to believe that they can do mathematics, because they don't have to wait to be shown an appropriate procedure by the teacher. It can help develop learners' confidence in their ability to do mathematics. As learners grapple to find the best strategy they will develop their reasoning skills and build higher order thinking skills.

Example 2

1 420 ÷ 20 = ?

Many learners quickly give the answer as 71. Different strategies can be used to calculate the answer.

Mmathapelo's calculation represents division as a fraction:

$$1420 \div 20 = \frac{1420}{20} = \frac{142\cancel{0}}{2\cancel{0}} = \frac{\cancel{142}^{71}}{\cancel{2}_{1}} = 71$$

Matthew's calculation involves working with whole numbers using division:

$$1420 \div 20 = 142\cancel{0} \div 2\cancel{0} = 142 \div 2 = 71$$

This is really the same method with a different layout. In both examples they first divided by 10 and then halved (or divided by 2). This strategy works because 1 420 and 20 are both multiples of 20: dividing by 10 and then dividing by 2 is the same as dividing by 20. It is the relationship between the numbers 1 420 and 20, that make this an efficient strategy.

Considerations for teaching

Whilst it is useful to encourage learners to develop their own calculation strategies, it is important that learners develop a strong enough number sense (which includes understanding the relationships between numbers) so that they choose efficient calculation strategies. Evaluating the most useful strategy to use in a particular situation or with a particular set of numbers is an important part of strategic competence.

Some calculation strategies are useful for some number ranges but become limiting with other number ranges. For example, if you don't know the answer to 8 + 3, you can **count on** 3 in ones *'9, 10, 11'*, but this strategy is not efficient when adding 2 948 + 7 426

Sometimes the most efficient strategy depends on the relationship between the numbers. For example, it is easy for learners to skip count to find the answer to some questions such as:

21 ÷ 7 — *7, 14, 21: the answer is 3*

or 100 ÷ 20 — *20, 40, 60, 80, 100; the answer is 5*

or 4500 ÷ 1500 — *1500, 3000, 4500; the answer is 3*

However, counting on in groups is not efficient for

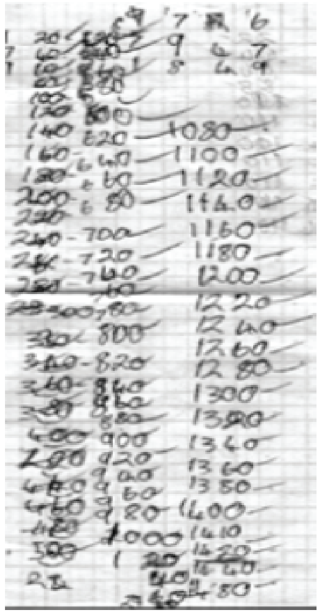
1 420 ÷ 20.

Alongside we show how a Grade 7 learner tried to solve

1 420 ÷ 20 by counting on in 20's (Scholar 2008). This takes a lot of time and there are many steps in which the learner could make a mistake. Counting on is not an appropriate strategy to solve 1 420 ÷ 20.

At other times the context in which a problem is posed, will impact on the choice of strategy or how to interpret the answer.

Learners need to develop a strong number sense and understanding of operations as the basis for developing their own calculating and problem-solving strategies.



Scholar, 2008.

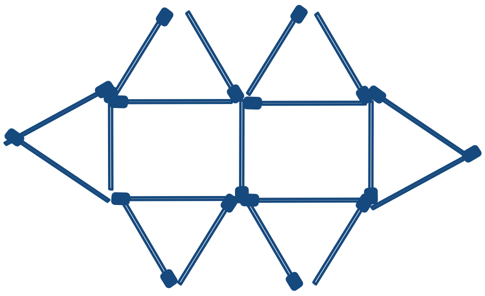
Learning benefits associated with learners being able to think out of the box

Using long division or even short division is not the most efficient way that you could calculate 1 420 ÷ 20. Example 2 illustrates some possibilities the individual thinking processes of learners when they are given freedom to experiment with their own strategies. In Example 2, learners drew on their understanding of numbers and the relationships between numbers. This develops their mathematical thinking.

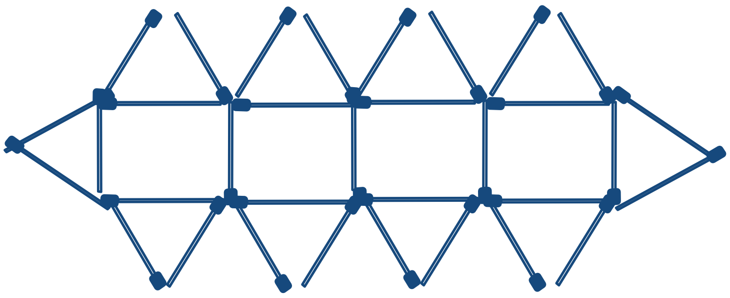
(d) Reasoning

Carefully look at the shapes below. They have been created with matchsticks. Shape number 2 and number 4 in a pattern made of shapes are given.

Shape number 2



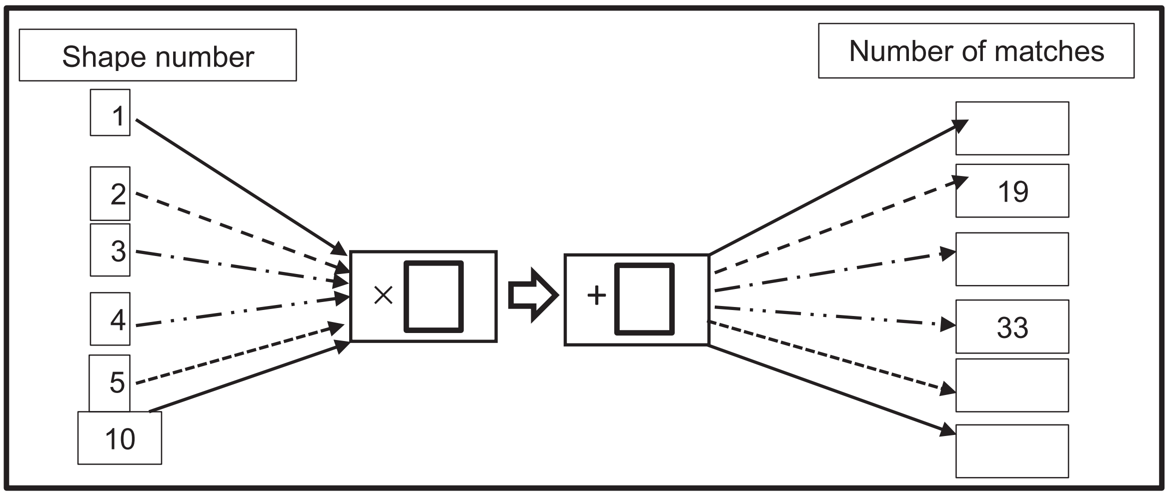
Shape number 4



1. Draw shape number 1, 2, 3 and 4 of the pattern (you can also use matches to build the shapes).
2. Explain in words how the pattern is growing.
3. Complete the table:

Shape number	1	2	3	4	5
Number of matches		19		33	

4. Now describe how the pattern is growing, by looking at the number of matches in each shape.
5. The flow diagram below helps you to find out what to do to the shape number to get to the number of matches. **There are two operations involved**. Complete the flow diagram.



Hint: Working with shape 2: find the number that you have to multiply by 2, and then add another number to get 19 (since 19 is the number of matches in shape number 2). See if that works for the other shapes in the pattern as well.

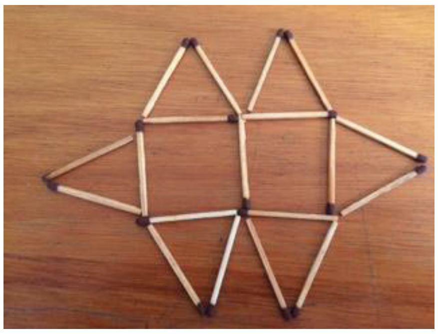
This example illustrates ***inductive reasoning***, using a growing number pattern. In this type of growing pattern, there is a relationship between the number of the shape in the pattern (also called term number), and the number of objects used to make the shape. In this example, the objects are matches.

We will discuss some learners' responses

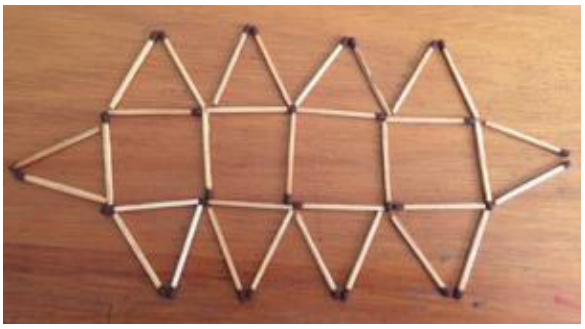
Kim packed out matches to copy shapes 2 and 4. She explains:

It is a pattern of squares and triangles. Shape 2 has 2 squares and 6 triangles around it. Shape 4 has 4 squares and 10 triangles around it. I count 19 matches in shape number 2 and 33 matches in shape number 4

Shape number 2



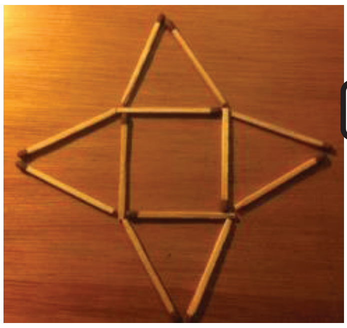
Shape number 4



She continues:

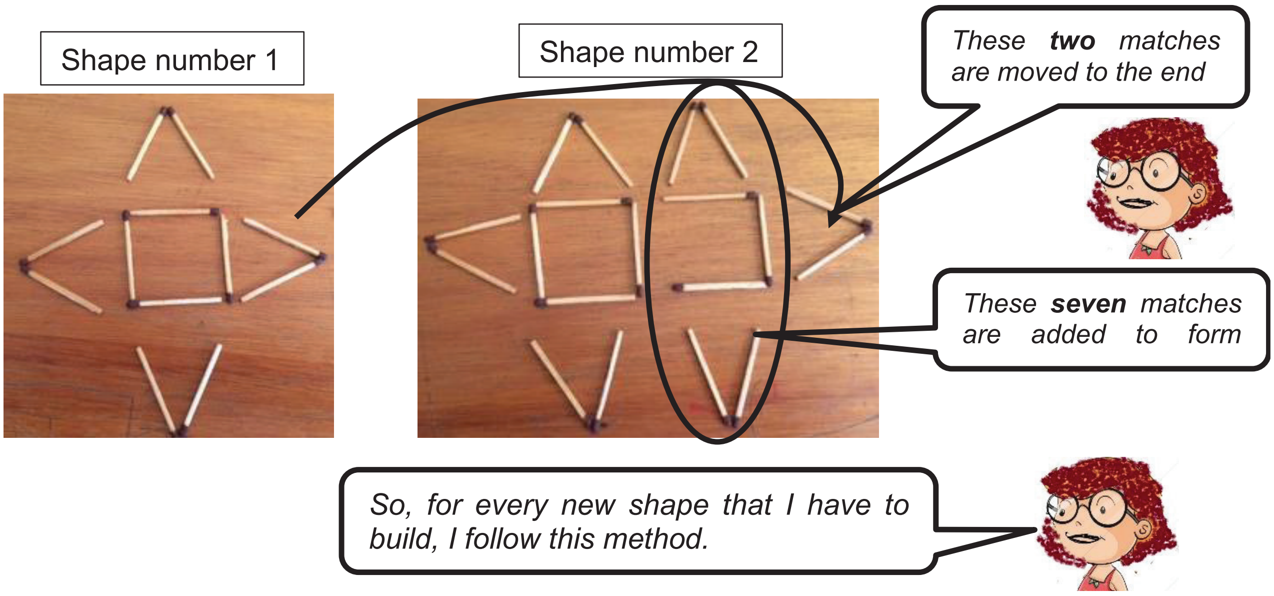
I am going to build shape 1 to see what it looks like. It has one square with 4 triangles around it.

She builds shape 1



I count 12 matches in shape 1

In her explanation to grow from shape 1 to shape 2, she moves the matches away from the shape and argues as follows:



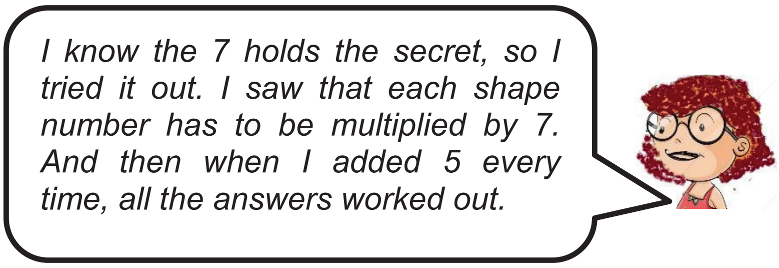
Kim completed the table:

Shape number	1	2	3	4	5
Number of matches	12	19	26	33	40

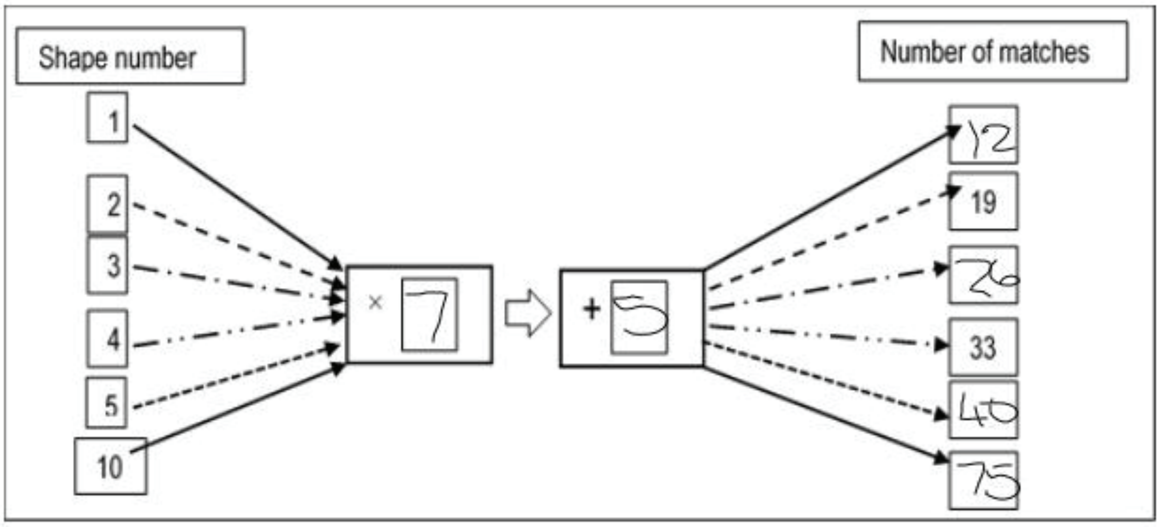
+7 +7 +7 +7

Pattern grows by 7

She moved on to the flow diagram:

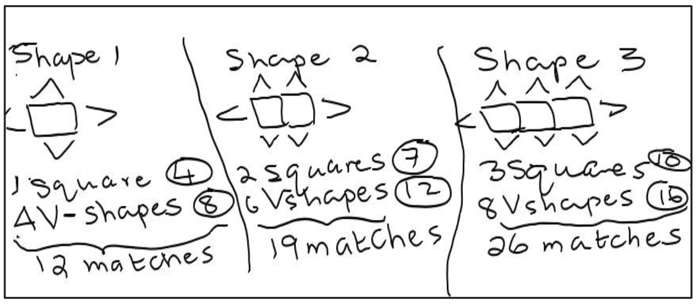


She moved on to the flow diagram:



Mogomotsi made drawings.

He explains:



I saw a pattern of squares in a row with V shapes around it.

To make the first shape I drew the square (4 matches) and then put a 'V shape' on each side. I counted 4 + 2 + 2 + 2 + 2 matches, 12 matches altogether. To make the 2nd shape I moved the one V on the right-hand side outwards and added 3 matches to make another square and a V above and below the new square. This took an extra 3 matches + 2 + 2 matches. This is 7 matches altogether. I can repeat this for each new shape. Each new shape needs 7 more matches.

Considerations for teaching

Explaining the pattern in words can sometimes be difficult. It is often easier to see what the pattern does, than to find the words to say how it works. Learners may use hand signals to help explain how they made new shapes in the pattern. Explaining what you see in words takes practice. Help to build learners mathematical language so that it is easier for them to explain their mathematical thinking. The use of language and reasoning go hand in hand.

The teacher can ask learners the following sorts of questions to help them explain their thoughts:

- 'What is the same about each shape?'
- 'How is the 4th shape different to the 2nd shape?'
- 'When you built the 3rd shape:
- How did you start?
- What did you do next?
- Why did you do that?'
- 'What if you move backwards from the 4th shape what would you take away to make the 3rd shape?
- 'What would you take away from the 2nd shape to make the 1st shape?'
- 'How does each shape differ from the one before?'

Focus learners both on what they see, i.e. the shapes in the pattern and on how they **built /drew** the shape, not only the shape itself.

Ask learners to use colours on the diagrams to show how:

- the 1st shape is contained in the 2nd shape.
- the 2nd shape is contained in the 3rd shape.
- they added the extra matches/ counters / dots / lines etc.

The teacher could also ask learners to leave spaces between the matches, or counters etc. to show where the preceding shape is contained in the current shape.

Train learners to listen to each other, even if they see the pattern differently.

Learning to understand the logic in other people's explanations is an important skill for teaching and learning. Not everyone thinks the same way. Not everyone sees pictures in the same way. Kim's and Mogomotsi's explanations show that they saw the differences between the shapes in different ways. People see many of the patterns in different ways; the process that people follow when they build or extend the pattern can be different, because of this their explanations will differ. Just because an explanation is not what you expected, it does not necessarily make it incorrect.

When young learners look at a pattern in a table, their first response is often to look at the difference between numbers in the bottom row of the table. For example, in the table below, learners may focus exclusively on the bottom row.

Shape number	1	2	3	4	5
Number of matches	12	19	26	33	40

Looking only at this row of numbers draws on the work that learners have done in skip counting and number sequences. So, in some ways it is logical that learners read the question as ___, 19, ___33, _____,

Learners may then see that they are adding 7 each time. This allows them to generate the next number in the pattern through counting. However, it does not allow them to predict the number of matches needed for any shape number. It does not help learners to see the relationship between 1 and 12; 2 and 19; 3 and 26; 10 and 75 etc. It does not help learners to generalise and predict beyond the numbers given.

Shape number	1	2	3	4	5
Number of matches	12	19	26	33	40

Using the function diagram helps learners to see the rule that they need to calculate how many matches they would use to build any shape number. Using the function diagram helps learners to see the relationship between the stage (shape number) and the number of matches.

Learning benefits associated with reasoning

One of the benefits of visual patterns like the example above is that it allows learners to make use of different solution strategies. This in turn allows for opportunities for learners to develop the skill of listening for and understanding the logic of the solutions of others. To delve deeper in the minds of your learners, the teacher will have to be prepared to ask learners to explain their understanding. The more teachers do this, the better they will be able to prepare mathematics lessons that address all learners' potential misconceptions.

In the example above learners generalise from two specific examples towards a general rule for making the patterns. They make use of inductive reasoning. Inductive reasoning is an important way for Intermediate Phase leaners to substantiate their arguments. The more learners practice using reasoning and justifying their answers, the better they will become at it. 'Logical thought' is the underlying factor that helps to make sense of mathematics.

In the Intermediate Phase learners state their general rule in words. Once learners have this general rule they can predict both what any shape number in the pattern will look like and how many matches they will need to build it. Being able to make predictions beyond the examples provided is important both in mathematics and in life.

3.4.3 Senior Phase

Senior Phase mathematics is characterised by several transitions and greater emphasis on properties, structure, and relationships between quantities. This is the phase in which learners are introduced to negative numbers, algebra and deductive reasoning. This means learners are expected to reason with definitions and properties of numbers and shapes, and this is a big shift for them and for their teachers.

(a) Conceptual understanding

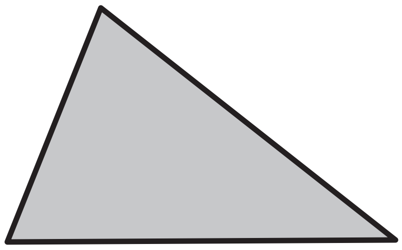
The example we have chosen to illustrate conceptual understanding focuses on the area of a triangle. In Grade 7 learners are introduced to calculating the area of triangles using a formula. However, when we ask learners in the Senior Phase what they understand by the term *area*, they often say 'length times breadth'. This suggests they do not fully understand the concept of area because they are referring only to the formula for the area of a rectangle and it's likely they don't even realise this.

The concept of area is important in mathematics and so our goal should be for all learners to make sense of the concept of area for any shape. When working with common shapes such as rectangles, triangles and circles, we want learners to relate the concept of area to the formulae.

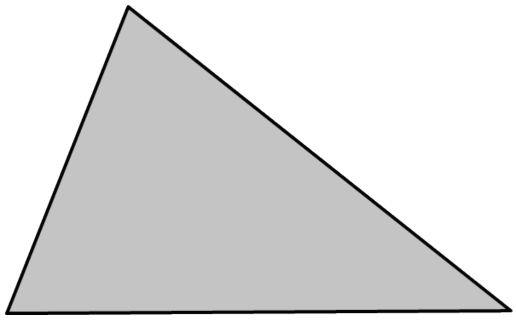
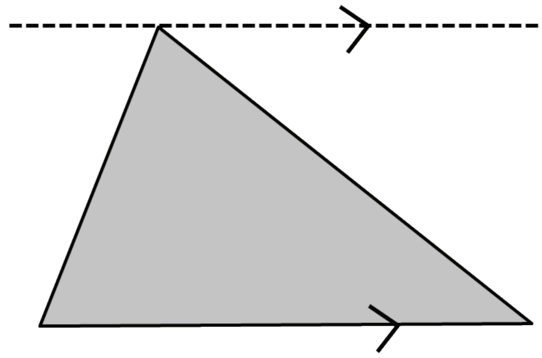
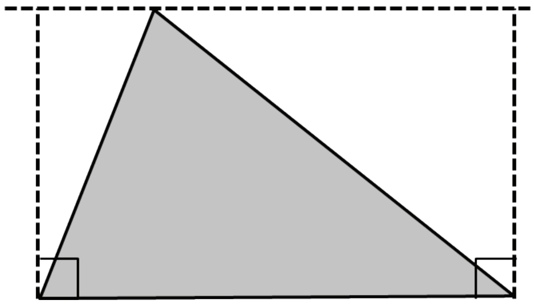
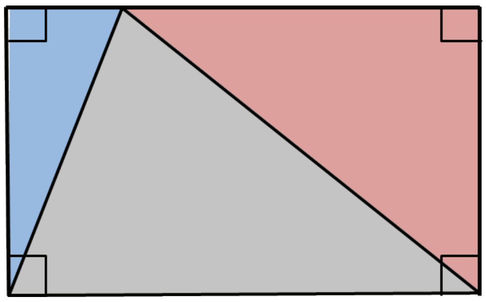
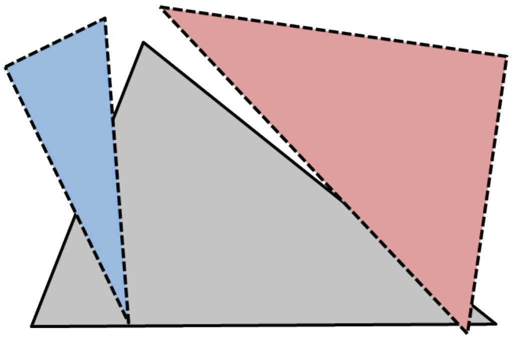
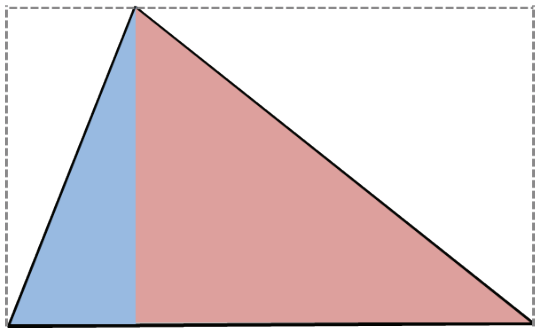
Example

The formula for the area of a triangle is Area = $\frac{1}{2}\, base \times height$.

Where does this formula come from?



The formula for the area of a triangle is derived from the formula for the area of a rectangle. So we will begin by drawing a rectangle 'around' the triangle and then cutting up the shape into three pieces. The task which is explained below is done in the form of a storyboard.

1. Draw a triangle with one side horizontal. We will call this the *base*.	2. Use a ruler to draw a line parallel to the base and passing through the third vertex of the triangle.
	
3. Draw two vertical line segments from the base of the triangle to meet the new line.	4. We now have a rectangle that contains three triangles.
	
5. Cut out the three triangles.	6. Place the two smaller triangles over the large triangle so that they cover it completely.
	

We can now see that the coloured triangles fill up the exact same space as the single grey triangle. This shows us that the grey triangle is half the area of the rectangle.

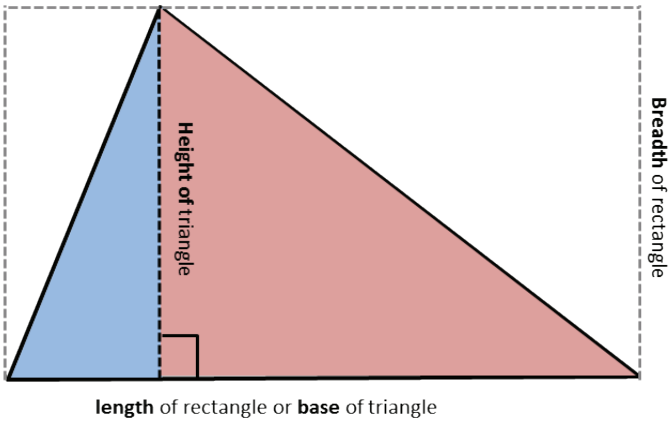
If we calculate the area of the rectangle as

$Area\ of\ rectangle = length \times breadth$, then we can calculate the area of the triangle as:

$Area\ of\ \Delta = \frac{1}{2}.length \times breadth.$

If we re-label *length* as *base* and *breadth* as *height*, then the formula becomes:

$Area\ of\ \Delta = \frac{1}{2}base \times height.$



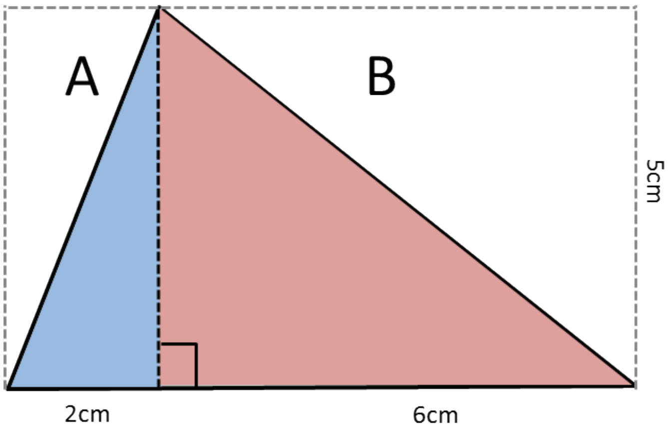
The six parts of the example illustrate a visual proof which might not be so easy for all learners to grasp. Some numerical examples could be helpful to illustrate the proof. Numeric examples do not build a proof or the general formula but they can be used to show patterns in results and to derive the formula.

In the diagram alongside the large rectangle has

$Area = length\ \times breadth$
$= 5 \times 8 = 40\ cm^2.$

We have 2 smaller rectangles, labelled A and B.

$Area\ of\ rectangle\ A$
$= length\ \times breadth$
$= 2 \times 5 = 10\ cm^2.$
$Area\ of\ rectangle\ B = length\ \times$
$breadth = 6 \times 5 = 30\ cm^2.$



The area of the blue triangle is clearly half the area of rectangle A, i.e. $5\ cm^2$

The area of the maroon/red triangle is clearly half the area of rectangle B, i.e. $15\ cm^2$

This means the area of the shaded triangle is $20\ cm^2$

Now if we use the formula for area of a triangle we can see that it also gives an answer of $20\ cm^2$:

$Area\ of\ \Delta = \frac{1}{2}.length \times breadth = \frac{1}{2}.8 \times 5 = 20\ cm^2$

Considerations for teaching

This example can be done with Grade 7, 8 or 9 learners .The intention is that the teacher will provide an opportunity for the learners to do the activity and will guide them as they work on it. The focus is on helping learners to see visually the relationship between the area of a rectangle and the area of a triangle. As can be seen, we have not focused much on calculations.

Learners need to see $length \times breadth$ as a single entity, not as a product of two separate numbers. We want learners to see the expression $\frac{1}{2}\ length \times breadth$, as 'half of the area', not 'half of the length multiplied by the breadth',

i.e. we want them to see $\frac{1}{2}(length \times breadth)$ not $\left(\frac{1}{2}\ length\right) \times breadth.$

Although the two expressions $\frac{1}{2}(length \times breadth)$ and $\left(\frac{1}{2}\ length\right) \times breadth$.

give the same answer, the **important idea** is that the triangle has half the area of the rectangle, not that you halve the base. We can see that the triangle and the rectangle have the same base; we are not halving the base of the triangle!

It is important to make links between the labels *length* and *breadth* of the rectangle, and *base* and *height* of the triangle. Learners must understand that the two line segments must be perpendicular to each other. In a rectangle, the length and breadth are always perpendicular to each other. In a triangle, when we refer to *height*, we always mean *perpendicular height* and *perpendicular* is always with reference to the base. This does not mean the base has to be horizontal and the height has to be vertical. It simply means there is a angle between the two line segments which are called the *base* and the *height* of the triangle.

Learning benefits associated with conceptual understanding

Learners with a conceptual understanding of the area of a triangle will know where the formula comes from and when to use it. They can make connections between the formula and the diagram. In time they will be able to make connections between the area of a triangle and the area of any quadrilateral. It is also more likely that learners will grasp why the base and the height must be perpendicular to each other. Consequently, learners will be less likely to use the lengths of non-perpendicular sides when calculating the area of a triangle. Learners will also be better able to calculate the area of complex shapes by breaking down the shape into familiar shapes like rectangles and triangles.

(b) Procedural fluency

Procedural fluency involves working accurately and efficiently when performing procedures. It also involves knowing when to use a particular procedure. In the example that follows, we offer a task to help learners to learn when to use a particular procedure.

The multiplication law of exponents states that:

“When we multiply powers with the same base, we add the exponents”

Use this law to simplify the following expressions if possible.

If you cannot use this law, try to simplify the expressions using another procedure you have learned if this is possible.

a) $m^3 \cdot m^4$

b) $m^2 \cdot m^5$

c) $m^2 + m^5$

d) $m^2 \cdot n^5$

e) $m^3 + n^3$

f) $m^3 + m^3$

Considerations for teaching

This set of examples requires learners to pay attention to both conditions of the law (multiplication and same base) for multiplication of exponents. These conditions are discussed individually below.

a) Law can be applied because we are multiplying and the bases are the same, answer is m^7

b) Law can be applied in the same way as (a) so answer is m^7

c) Law cannot be applied because we are not *multiplying* the powers. These are not like terms so cannot be simplified further.

d) Law cannot be applied. Although we are multiplying powers, the *bases are not the same*. Cannot simplify further.

e) Law cannot be applied. Firstly, we are not multiplying powers. Although the exponents are the same, these are not like terms therefore they cannot be added. However, some learners might want to write $(m+n)^3$ which is not correct.

f) Law cannot be applied. The bases and the exponents are the same they are *not being multiplied*. However, they are like terms and so can be added to give $2m^3$.

Learning benefits associated with procedural fluency

Learners who can execute mathematical procedures fluently know *when* a procedure is appropriate. They are also better able to work on more challenging tasks without battling to execute the procedures that form part of the solution.

(c) Strategic competence

Strategic competence involves choosing appropriate strategies to answer a given mathematical task. Often learners can choose from a range of strategies they have been taught. However, there are also instances when learners will need to devise their own strategy to solve a problem.

We provide two examples to illustrate strategic competence in the Senior Phase. Example 1 focuses on familiar content involving graphs. Example 2 is a problem solving type question.

Example 1 requires learners to make connections between different representations. When learners make use of a particular strategy, they reveal which representations they are paying attention to and how they make connections between the different representations. In the case of the linear function (or straight line graphs as they are referred to in Grade 9), learners need to be able to show how the values of and in the standard equation $y=mx+c$ relate to the graph. In other words they need to show that c-value gives the y-intercept and that the m-value is the gradient of the graph.

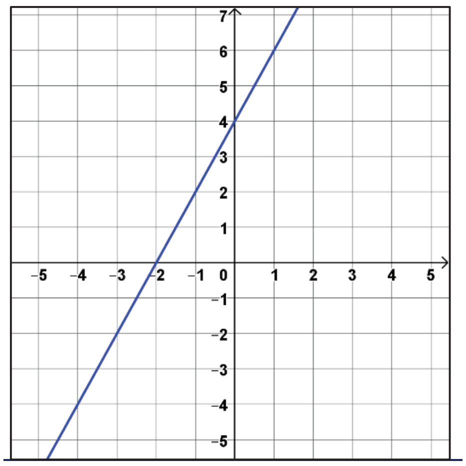
Example 1

Here are three equations and a graph of a straight line.

$$y = 2x + 4$$
$$y = -2x + 4$$
$$y = x + 4$$

Which equation represents the graph?

Justify your answer.



Below we describe five different strategies that learners might use to decide which equation represents the graph. It is likely that learners will have been taught four of the strategies before they encounter this question.

1) **Strategy 1: Reasoning by elimination/inspection**

Learners might say that $y = -2x + 4$ cannot be the correct equation because $m = -2$ which is a negative gradient but the graph has a positive gradient. They might then say the gradient of the graph is not 1 because the x- and y-intercepts are different, i.e. if the gradient were 1, then the graph should cut the x- and y-axes at the 'same' value/number. This leaves $y = 2x + 4$ as the only possible answer.

2) **Strategy 2: Using equation of graph to find m**

Learners may focus on the fact that they know the y-intercept is 4 and that this is the c-value in the equation $\mathrm{y} = \mathrm{mx} + \mathrm{c}$.

They can then substitute the coordinates of a point, and solve for . For example, if they substitute (2 ; 0):

$$\begin{aligned} 0 &= \mathrm{m}(-2) + 4 \\ 2\mathrm{m} &= 4 \\ \mathrm{m} &= 2 \end{aligned}$$

They can choose any other point on the line but not (0 ; 4). For example, if they choose (1 ; 2):

$$\begin{aligned} 2 &= \mathrm{m}(-1) + 4 \\ -2 &= -\mathrm{m} \\ \mathrm{m} &= 2 \end{aligned}$$

Now learners can conclude that $y = 2x + 4$ is the correct equation.

3) **Strategy 3: Using the gradient formula**

Learners could substitute the coordinates of any two points on the line into the gradient formula and show that the gradient is 2. Most learners are likely to choose the intercepts with the axes (2 ; 0) and (0 ; 4):

$$\mathrm{m} = \frac{4 - 0}{0 - (-2)} = \frac{4}{2} = 2$$

However, learners can choose any two other points on the line to determine the gradient. For example they might choose (1 ; 6) and (3 ; 2) because these are easy to read from the grid.

$$\mathrm{m} = \frac{6-(-2)}{1-(-3)} = \frac{8}{4} = 2$$

Now learners can conclude that $y = 2x + 4$ is the correct equation.

4) **Strategy 4: Using ‘rise over run’**

Learners could use the idea of ‘rise over run’ to show that the gradient is 2. For example they could start at (– 2 ; 0) and move up 4 units, then right 2 units to end at (0 ; 4). Thus $\frac{\text{rise}}{\text{run}} = \frac{4}{2} = 2$. Alternatively, learners could start at (0 ; 4), move down 4 units to (0 ; 0) and then left to (2 ; 0). Here $\frac{\text{rise}}{\text{run}} = \frac{-4}{-2} = 2$ They could even start at (– 4 ; – 4) then move up 2 units and across 1 unit to reach the line again at (3 ; 2). So $\frac{\text{rise}}{\text{run}} = \frac{2}{1} = 2$ Using any of these, they can conclude that $y = 2x + 4$ is the correct equation.

5) **Strategy 5: Reading off the intercepts**

Learners may say that the equation of the given graph is $y = -2x + 4$ because they see the graph cutting the x-axis at –2 and the y-axis at 4, and they see these two values in the equation too. This is an incorrect strategy.

Considerations for teaching

We begin with a discussion of each of the five strategies.

Strategy 1: The learner has made a connection between the sign of and the slope of the graph. Then she has recognised that if a graph has a slope of 1, it will cut the axes at the ‘same number’. In the case of $y = x + 4$, the y-intercept is 4 and the x-intercept is – 4. So it is not correct to treat these as the ‘same’ number/value because the signs are different. It may be useful to point out that that the distance from the origin to each intercept is the same and the slope of the line is $45°$. Note that this learner has not actually calculated the gradient of the given graph.

Strategy 2: This learner has connected the constant in the equation with the y-intercept of the graph. This is a useful strategy provided that the y-intercept of the graph is known. Learners should also make sure that the equation is in standard form $(y = mx + c)$ otherwise the constant may not be the y-intercept (e.g. if the equation were given as y+x+4+=0. Note that we can substitute the coordinates of any point to find m, except for the coordinates of the y-intercept. We cannot use the coordinates of the y-intercept because by replacing c with 4, we have actually already substituted the coordinates of the y-intercept and so we can’t use the same point again.

Strategy 3: When working with the gradient formula, it is important for learners to understand that they can choose any two points that lie on the line. Often the coordinates of the intercepts are easy to work with and so they are a good choice to keep the calculation simple. However, learners often make errors with negatives. Therefore they should check that their answer makes sense in relation to the graph, e.g. if they calculate the m-value to be negative but the graph has a positive slope, then they have made a mistake in their calculations. It will also be useful to show that even if we choose different points, we still get the same gradient.

Strategy 4: The ‘rise over run’ approach focuses on the graphical representation. It works with the difference in the x-coordinate and the difference in the y-coordinate of the chosen points. Learners need to know the conventions that apply here: movement up or right is associated with a positive number, while movement down or left is associated with a negative number. We have shown three examples in order to emphasise that the starting point can be changed. This also shows that movement up and then right is the same as movement down and then left. Note that we don’t need to start on the axes when working out the rise and run. However, we must start on the graph and end on the graph.

Strategy 5: Learners often focus too much on visual aspects in questions involving graphs and equations. Here the learner is linking up the values of the intercepts with the m-value and c-value in the equation. This is an inappropriate strategy and an incorrect solution. In order to identify the correct equation, learners must understand the meaning of the m-value and c-value in the equation.

Once learners have correctly identified the equation, they could be invited to draw the graphs of the other two straight lines. This requires them to move from an algebraic representation to a graphical representation. They will see that all three equations have the same y-intercept.

Learning benefits associated with Strategic competence

The strategies described above reflect the different ways of approaching this task. Strategies 1 to 4 provide different approaches that draw on different procedures. It will be helpful for learners to see each of these strategies and then to choose the one/s that they feel most comfortable with. Ideally all learners should be able to use more than one strategy to answer this question.

Strategy 5 shows that the learner does not fully understand the relationship between the equation and the graph. This is helpful information for the teacher because it provides an indication that learners are looking for inappropriate visual cues when attempting this task.

The next example is one where learners need to think of ‘out of the box’ in order to find the solution. The task involves only whole numbers but the focus is on learners’ developing appropriate strategies and reasoning. This example shows clearly the close links between strategic competence and reasoning.

Example 2

Each shape represents a number. Same shapes represent the same number. The number next to a row or column gives the total for the row or column. What is the missing number?

▲	■	■	▲	28
✕	■	✕	■	30
●	▲	●	●	18
●	■	●	●	20
?	30	23	22	

Solution:

6	8	8	6	28
7	8	7	8	30
4	6	4	4	18
4	8	4	4	20
21	30	23	22	

Considerations for teaching

We begin with a discussion of 3 possible strategies that learners may adopt. We have added labels to the rows and columns for easy reference in this discussion. Learners need to decide where to start. Different strategies will have different starting points We discuss 3 possible strategies:

	C1	C2	C3	C4	
R1	▲	■	■	▲	28
R2	✕	■	✕	■	30
R3	●	▲	●	●	18
R4	●	■	●	●	20
	?	30	23	22	

Strategy 1: Focus on column 2: 3■+▲=30. Assume that ■=9, then ▲=3. We can use row 1 to check: 2■+2▲= 2(9)+2(3) but this gives 24, not 28 as required. Therefore there is an error in strategy 1.

Strategy 2: Focus on rows 3 and 4: We have 3●+▲=18 and 3●+■=20. So we can deduce that ▲+2=■. Now in row 1 we'll have: ▲+(▲+2)+(▲+2)+▲=28. This means that 4▲=24 So ▲=6 and ■=8. Now in column 2, we have 3■+▲= 3(8)+6=30 which suggests we are correct so far.

Strategy 3: In row 4 we have 3●+■=20. Let's assume that each ●=5, then ■ must also be 5. This is not possible because 2 different shapes can't have the same number. So there is an error in strategy 3.

We suggest that learners work individually on this task so that they can select their own strategies. Thereafter they can work in pairs to compare answers and strategies. It is important to ask learners to explain their strategies publicly. It is likely that many learners will start with one strategy and then abandon it or revise it when they realise it doesn't work. However, when they report their strategies, they might not talk about the 'false starts'. It is important for the teacher to encourage learners to share about strategies that didn't work and to explain how they realised that the strategy didn't work.

Learning benefits associated with Strategic competence

We have described 3 strategies for approaching this task. As can be seen, 2 of the strategies have errors in the reasoning because either the relationship in a row/column does not hold or because 2 shapes have the same number which does not satisfy the conditions of the problem.

In order for learners to develop strategic competence, they need to test whether their strategies are suitable for the problem, and whether they produce a suitable answer. This involves some form of mathematical reasoning which shows the close relationship between these two dimensions of the framework. Choosing and then rejecting inappropriate or inefficient strategies is part of the learning associated with developing strategic competence.

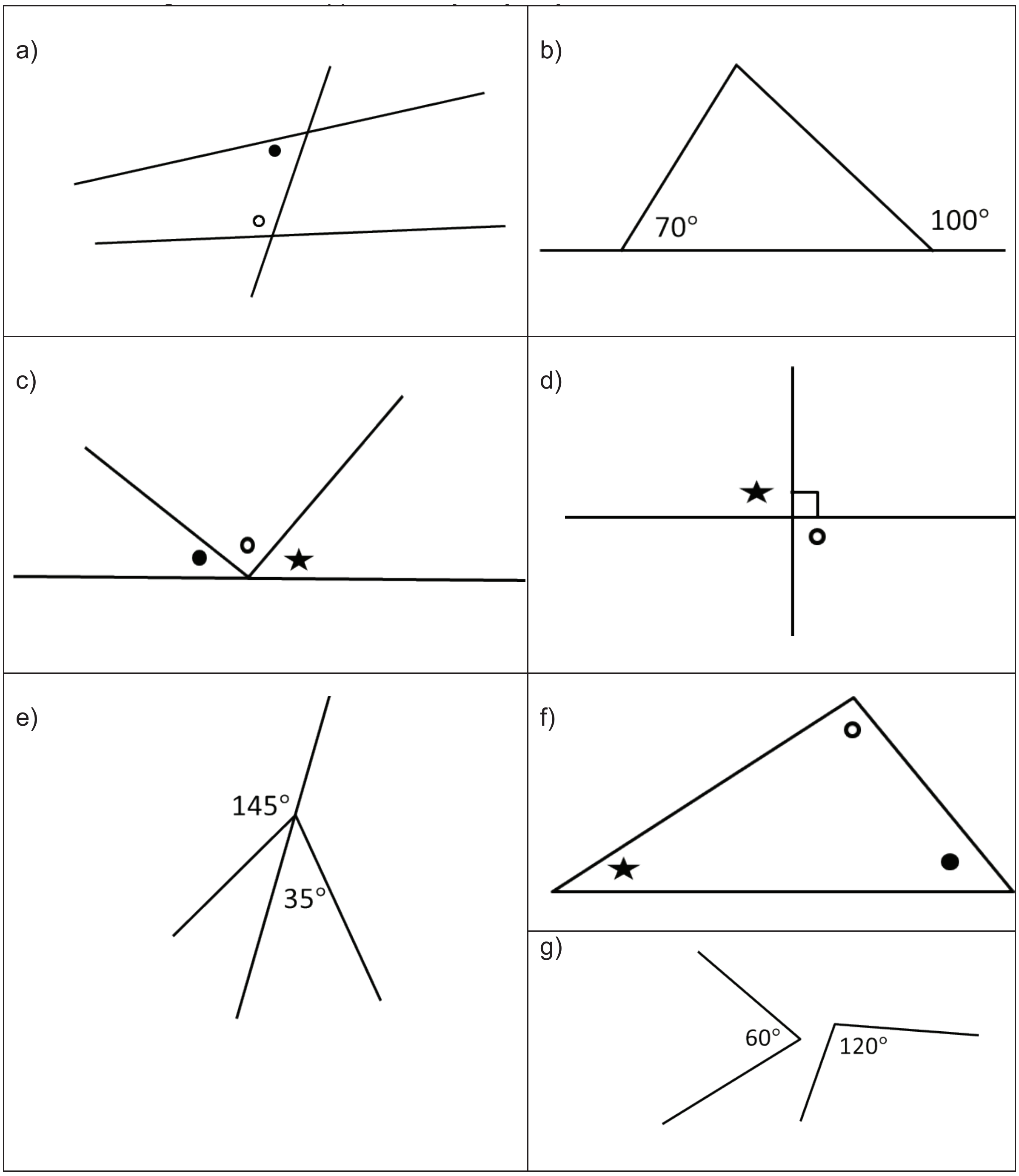
(d) Reasoning

In the Senior Phase learners start to reason with properties and definitions. For example in geometry, it is no longer sufficient to focus on what a shape looks like, they need to focus on the properties of shapes. The example below involves supplementary angles. The definition of supplementary angles has 2 criteria: exactly 2 angles, and the angles must sum to However, we know that many people think that supplementary angles must be adjacent. We also know that many people don't realise that supplementary angles involves only 2 angles. This task requires learners to pay attention to both aspects of the definition to justify their answer.

Supplementary angles are defined as follows: *Two angles are supplementary when they add up to* $180°$.

In each diagram below, two or more angles are indicated with symbols like ● ★ ○ or specific sizes.

- Which diagrams show supplementary angles?
- If the angles are not supplementary, say why.



Solution:

a) **Not supplementary.** These 2 angles are co-interior because they lie between a pair of lines and on the same side of the transversal. Learners often think that all pairs of co-interior angles add up to $180°$. But co-interior angles only add up to $180°$ if the lines are parallel. In this diagram the lines are not parallel. Therefore the angles are not supplementary.

b) **Not supplementary.** There are two angles but they do not add up to $°$,

i.e. 70° + 100° = 170° .

c) **Not supplementary.** There are three angles that lie on a straight line. Therefore the angles add up to $180°$ But the angles are not supplementary because there are three angles and supplementary refers to only two angles.

d) **Supplementary.** This is a more complex example than the others. There are three angles indicated in the diagram. One of the angles is 90° which means the two lines are perpendicular to each other. This means all three angles are 90°. We can deduce that

★ + 90° = 180° because they are angles on a straight line. Similarly, we can also deduce that 90+○ = 180° because they are angles on a straight line. Now we can conclude that ★ + ○ = 180°. This means the two angles are supplementary.

e) **Supplementary.** There are two angles that add up to 180°, i.e. 145° + 35° = 180°

f) **Not supplementary.** There are three angles in a triangle and so the angles add up to 180But, as in the earlier example, the angles are not supplementary because there are three angles.

g) **Supplementary.** There are two angles that add up to 180, i.e. 60° + 120° = 180° . The angles do not need to be connected by a common vertex or have a common arm.

Considerations for teaching about supplementary angles

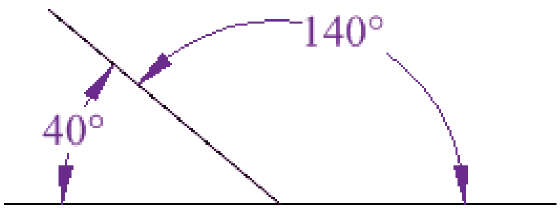
In this task learners need to see that there are two aspects to the definition of supplementary angles: (a) there must be **exactly** two angles (no more, no less) and (b) the angles must add up to 180°.

The seven diagrams require learners to focus on both aspects of the definition. There are two examples where the sum of the angles is 180° but there are three angles in these examples. This means the angles are not supplementary.

Learners often assume that the angles must be adjacent (i.e. next to each other). But the definition does not say this. The definition says nothing about the position of the angles in relation to each other. So **any** two angles that add up to 180° are supplementary.

It is not surprising that learners think the angles must be adjacent because text books often provide diagrams like the one alongside to illustrate supplementary angles.

While the diagram does indeed provide an example of supplementary angles, it shows a specific case and this can lead to the misconception that supplementary angles must be next to each other and form a straight line.



We expect learners to know the definition of supplementary angles. However, knowing the definition doesn't mean that they can correctly answer the task posed here. It is therefore important to expose them to tasks that require them to reason with both aspects of the definition.

Learning benefits associated with the ability to reason

If learners can reason mathematically, they can apply definitions, laws, theorems etc. to solve problems. They do not need to learn things off by heart. They can also reason to decide whether an answer is correct or whether another person's argument makes sense. This makes them more independent as mathematical thinkers. They no longer have to depend so much on their teacher to tell them whether or not an answer is correct.

3.4.4 FET band

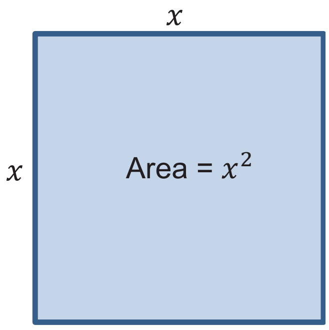
As mentioned elsewhere, the four dimensions in the framework are not mutually exclusive. Most mathematical problems will involve elements of all four dimensions. Although we present one example to illustrate each of the dimensions, these examples will also contain elements of the other dimensions.

(a) Conceptual understanding

Mathematics is a web of different ideas and concepts and relations between them. Conceptual understanding involves proper understanding of these concepts as well as the relations between the different concepts. Learners sometimes do what is called "mindless symbolic manipulation" without really understanding why and what we are doing. A good example of this is when we ask them to "complete a square". The "mindless" recipe is to *add the square of half the x-coefficient!* WHY do we do this – because it works? In the first example we show how to develop a conceptual understanding of this process.

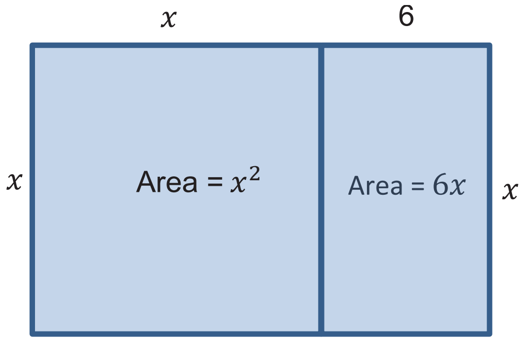
Example 1: What must be added to the expression $x^2 + 6x$ to make it a perfect square?

Let us illustrate the expression in a visual way. We know that a square with side lengths of x, has an area of x^2:



Our problem is that if we have an expression like $x^2 + 6x$ it is not a square anymore. It is a rectangle with an area of $x^2 + 6x = x(x+6)$.

The side lengths of the rectangle are therefore x and $x + 6$ and. We can represent the expression in the diagram as follows:

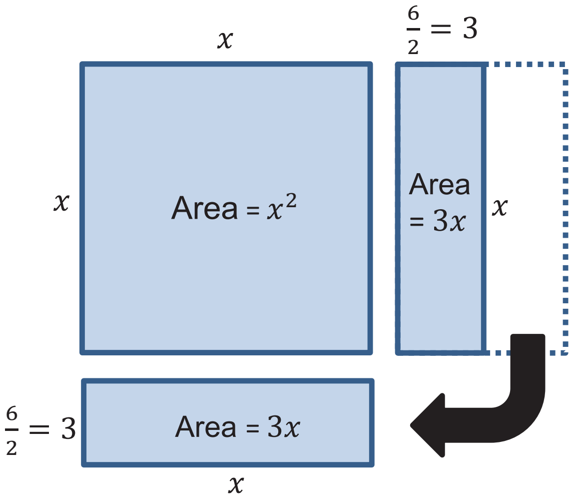


We have a new shape (a rectangle) which is made up of two shapes:

- a square with an area of $x \times x = x^2$
  plus
- a rectangle with an area of $6 \times x = 6x$.

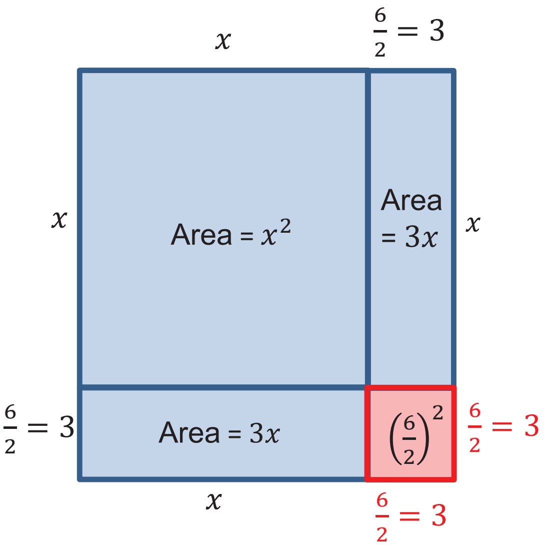
We want to investigate what we can do to the rectangle to make it a square.

We can divide the smaller rectangle of 6 by x into two smaller rectangles, with an area of $3x$ each. This is illustrated in the next diagram:



Moving and rotating the one half to the bottom we almost create a square. There is just a small square-shaped opening on the bottom right of the new shape.

We need to add the little red square in the bottom right corner to make the bigger shape into a perfect square. This is shown below. We now need to work out the dimensions and the area of this little red square.



The area of the small red square required to complete the large square is $\frac{6}{2} \times \frac{6}{2} = 3 \times 3 = 9$. Thus if we add 9 to the expression $x^2 + 6x$, we have $x^2 + 6x + 9$, which we have shown above is a perfect geometric square. Algebraically we can factorise this expression as follows:

$x^2 + 6x + 9 = (x + 3)(x + 3) = (x + 3)^2$. So $x^2 + 6x + 9$. So is a perfect square trinomial!

There is an important relationship between the 6, the coefficient of x, and the number that we add to make the expression a perfect square. To complete the square of the expression such as $x^2 + 6x$ we add a small square with an area of $\left(\frac{6}{2}\right)^2 = 3^2 = 9$.

It is therefore possible to complete the square of the expression $x^2 + 6x$ without a diagram.

You can follow the following steps:

Step 1: Determine the coefficient of x : $+6$

Step 2: Divide the coefficient of x by 2: $\frac{6}{2} = 3$

Step 3: Square the previous answer: $\left(\frac{6}{2}\right)^2 = 3^2 = 9$

This gives you the term that needs to be added to the expression to make it a perfect square: $x^2 + 6x + 9$

Considerations for teaching

Learners may use different routes to reach the conclusion that to get a square you 'add the square of half the coefficient of the middle component'. This exemplar demonstrates that physical demonstrations of this conclusion can be used to enhance the conceptual understanding of what is sometimes taught purely procedurally. The fact that the learners had to 'verbalise their observations' facilitates their conceptual understanding as well as their procedural fluency.

Some learners find visual concepts (like area) difficult to understand. It may be necessary to revise these concepts before building on them. The example shown above may be used in a powerful way to develop the initial idea of completing the square (using visual concepts to build algebraic understanding).

Teaching for understanding implies teaching in such a way that learners will know the conceptual basis for the procedures that they must learn. Learners should not be happy to know something without knowing where it comes from. Interpreting quadratic expressions using the area of rectangles (as in the example above) should not feature in all of the teaching about completing the square but teachers should use this example at least once to show learners where the words 'completing the square' come from. This will help dispel the myth that maths is a mysterious game invented by someone, somewhere.

The rule associated with completing the square is easy to visualise and demonstrate when b (the coefficient of the term) is positive. It is also applicable in the case where b is negative. Teachers could thus use the illustration to demonstrate the concept in the case of positive x coefficients and then generalise the rule to expressions with negative x coefficients.

This approach has limitations when it comes to more advanced quadratic expressions but it can be used to help learners to make the jump from a conceptual explanation of a process to a procedural approach.

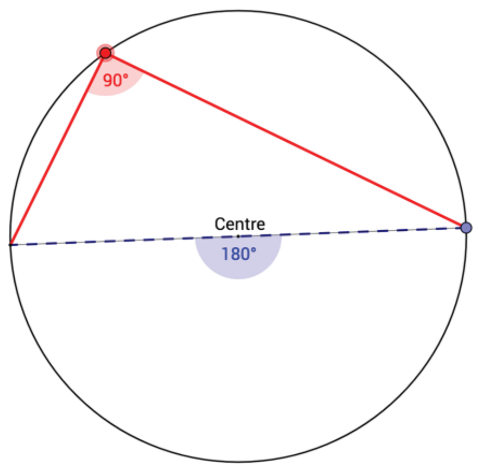
Learning benefits of teaching for conceptual understanding

Completing the square algebraically has little meaning for learners if it is only done procedurally. The benefit of this particular visual representation is that it enables learners to see how completing the square can be visualised. It gives meaning to the expression 'completing the square'.

Why do learners need to be able to complete the square? They use it later on to determine the minimum or maximum value of a quadratic expression. It will also enable them to sketch the graph of a parabola. Completing the square is a useful algebraic skill.

Example 2:

The angle on the circle subtended by the diameter is always 90°



This activity could be done with or without the use of ICT. The following steps could be used by the teachers for learner activities:

Without ICT:

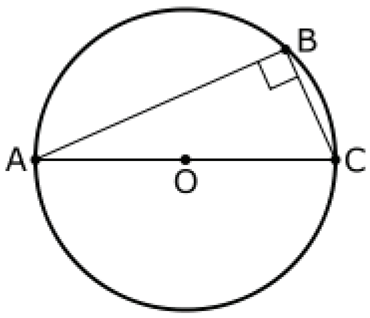
- Using your compass, draw a circle of any radius.
- Using your ruler, draw a diameter of the circle (a diameter goes straight across the circle, through the centre to the edge of the circle).
- Construct any angle on the circumference of the circle from the two endpoints of the diameter.
- Measure the angle; what do you notice?
- Compare your result with that of your friends.

Using ICT:

- Open the sketch:

  https://school-maths.com/applets/Circle_Geometry.html#material/61550

- Drag the red dot.



- What did you notice about the size of the angle on the circle?
- Ask the learners to formulate a conjecture. It can be an informal conjecture, e.g. this angle on the circle is 90°

Learners have to make the transition towards a more mathematical formulation:

If A, B, and C are distinct points on a circle where the line AB is a diameter of the circle, then the angle ∠ACB is 90°. We can therefore conclude that the diameter of a circle subtends a right angle to any point on the circle.

Considerations for teaching

Giving the learners the opportunity to discover things and to describe their own discoveries, using their own words, will help them with the development of their mathematical language. In this example learners have to use words like diameter, circle, subtends, and perpendicular.

Learning benefits of teaching for conceptual understanding

The effective use dynamic software and Apps (e.g. GeoGebra and Desmos) has the potential to strengthen learners' conceptual understanding. In the classroom, dynamic geometry can develop and reinforce concepts, enrich visualisation, and rectify misconceptions. By clicking and dragging the cursor, learners can discover relationships, and formulate and test conjectures. This inductive process lays the foundation for justification as well as explanation by deductive proof.

(b) Procedural fluency

Although it is important to understand deeply and conceptually, once this is accomplished, a learner need not do this conceptual thinking every time he/she has to use a procedure. Once the procedure is understood conceptually, you use it as a standard procedure. To conduct mathematics one needs to be fluent in using these procedures. On the other hand, having used a procedure repeatedly often enhances conceptual understanding of the concepts and procedures involved.

Example 1: What must be added to $x^2 - 8x$ to write it as to write it as $(x - \blacksquare)^2$.

In the previous example, we completed the square of $x^2 + 6x$ by using a diagram.

To determine this number you can follow these steps:

- **Step 1**: Determine the coefficient of the term: 6
- **Step 2:** Divide the coefficient of the term by two: $\frac{6}{2} = 3$
- **Step 3**: Take the square: $\left(\frac{6}{2}\right)^2 = 3^2 = 9$
- **Step 4:** Adding this term to $x^2 + 6x$ we get $x^2 + 6x + 9 = (x + 3)^2$

We can follow the same procedure to complete the square of $x^2 - 8x$:

- **Step 1**: Determine the coefficient of the x term: -8
- **Step 2:** Divide the coefficient of the x term by two: $\frac{-8}{2} = -4$
- **Step 3:** Take the square: $\left(\frac{-8}{2}\right)^2 = (-4)^2 = 16$
- **Step 4:** Adding this term to $x^2 + 6x$ we get $x^2 - 8x + 16 = (x - 4)^2$

We can write $x^2 - 8x + 16$ as $(x-4)(x-4) = (x-4)^2$. Mathematicians call this a complete square.

Considerations for teaching

Learners often make the following mistake when they square negative numbers:

$\left(\frac{-8}{2}\right)^2 = -(4)^2 = -16$ or $(-4)^2 = -16$.

To help them to avoid this mistake it may be helpful to write $(-4)^2 = (-4)(-4) = +16$.

Completing the square is used often in algebra to find minimum and maximum values. Like in the case of finding the turning point of the parabola using the formula $y = a(x-p)^2 + q$. Completing the square of the expression $ax^2 + bx + c$ enables us to rewrite it as $a\left(x + \frac{b}{2a}\right)^2 + \frac{b^2-4ac}{4a}$, where the maximum or minimum value is $\frac{b^2-4ac}{4a}$.

Learners need to be able to complete the square in equations as well. In the case of the equation $x^2 - 4x + 1 = 0$ teachers should explain to learners that they need to use the additive inverse of 1 both sides to change the form of the equation: $x^2 - 4x = -1$. Learners can then follow the above mentioned steps to complete the square, however since they are working on an equation, they have to add both sides: $\left(\frac{-4}{2}\right)^2 = 4$ both sides: $x^2 - 4x + 4 = -1 + 4$.

Learning benefits of teaching Procedural Fluency

To become a fluent mathematician, a high level of procedural fluency should be acquired - it is part of disciplinary knowledge – one cannot develop everything from scratch every time.

Mastery of certain skills is essential. The procedural skill shown in this example is that of ‘completing the square’.

Completing the square is used often in Algebra to find minimum and maximum values and is thus a useful skill.

Example 2: Factorise $x^2 + 6x + 8$

When you **multiply** two expressions the answer is called the product. When you **factorise** an expression, you write it as the product of two factors.

To determine the product of $(x+4)$ and $(x+2)$ we can use the distributive property:

$$\begin{aligned}(x+4)(x+2) &= x.(x+2) + 4(x+2)\\ &= x.x + x.2 + 4.x + 4.2\\ &= x^2 + 6x + 8\end{aligned}$$

Conversely, to factorise the expression $x^2 + 6x + 8$, we must write it as the product of two factors. The procedure to factorise $x^2 + 6x + 8$ is as follows:

Find the two factors of 8, the constant, which when added, give you 6 (the coefficient of the x term).

$x^2 + 6x + 8$
$= x^2 + 4x + 2x + 8$
$= x(x+4) + 2(x+4)$ … take out the common factor
$= (x+4)(x+2)$

Considerations for teaching

There are many different factorisation procedures that should be taught. Learners must know these procedures and know which procedures are best to use for a particular task. This means that learners need to have more than one procedural strategy to draw on. Teachers are responsible for the teaching of all of these procedures so that learners will be able to use increasingly efficient procedures.

Learning benefits

Learners have to be fluent in factorising quadratic expressions. It is essential when solving quadratic equations, such as
$x^2 + 6x + 8 = 0$
$(x+4)(x+2) = 0$
$x + 4 = 0$ or $x + 2 = 0$
Therefore $x + 4 = 0$ or $x + 2 = 0$.

(c) Strategic competence

It is important that learners be able to decide on appropriate procedures to solve mathematical problems – especially when it comes to a modelling problem ("word sum"). Sometimes one has to think carefully about the problem and often the learner has to think "out of the box" and come up with his/her own procedure or approach to solve the problem.

In this section we present a problem that deals with the concept of area. The problem is divided into two parts – the first one is a closed problem with one solution (the dimensions that yield the maximum area) the second one is open ended (it depends on the dimensions under consideration).

The discussion shows that even if there is one solution to a problem there may be several strategies used to find this solution.

Example 1: You have 24 m of fencing to enclose a rectangular garden.

Closed problem: What are the dimensions of the garden that will yield a maximum area?

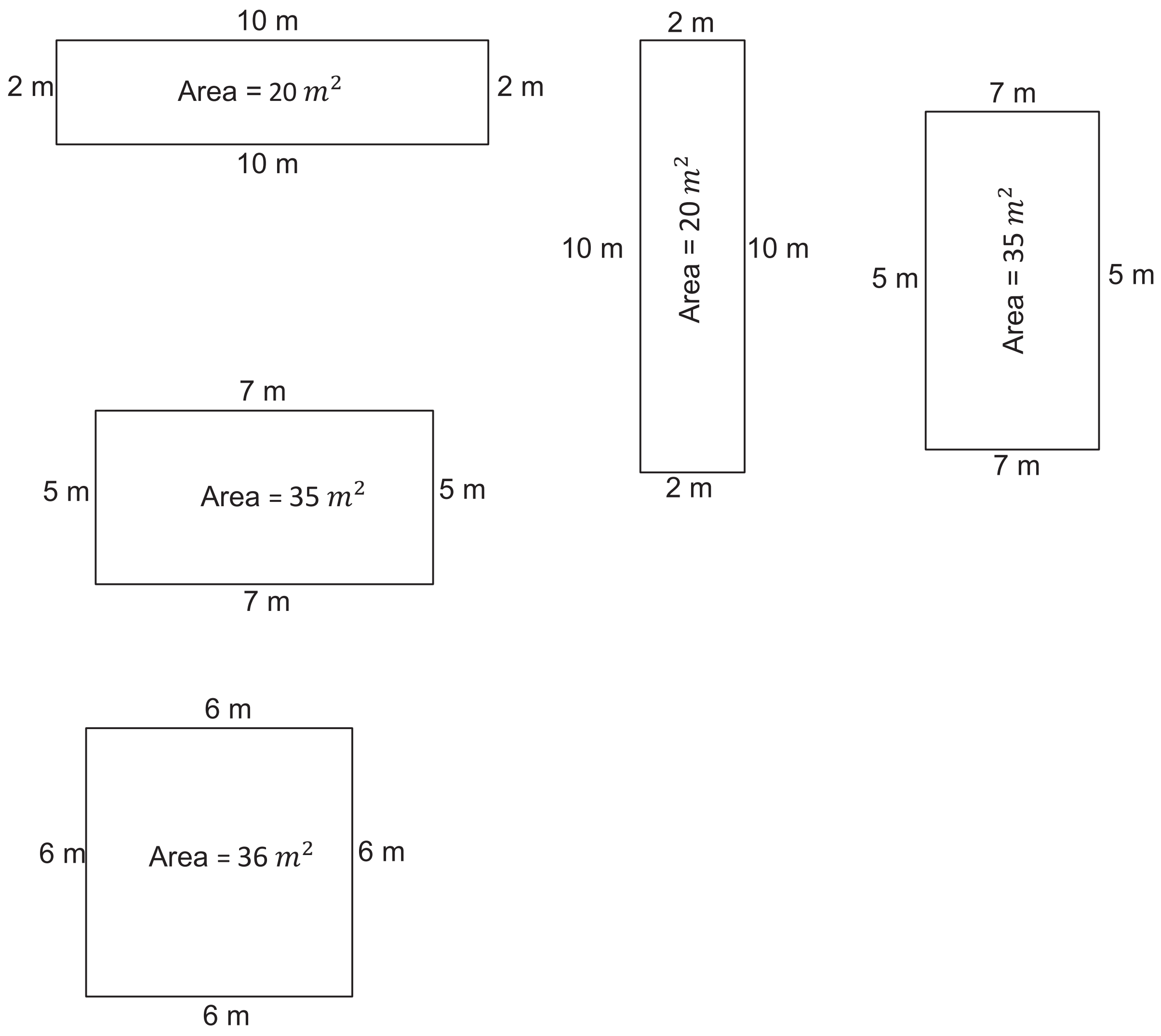
Open ended: What is the area of the garden?

Learners can use different methods to solve this problem, e.g. trial and error, identify the turning point of the graph of a parabola, completing the square using the formula

, $y = a(x-p)^2 + q$ using $x = -\frac{b}{2a}$, or using calculus $f'(x) = 0$ to identify the maximum value.

Strategy 1: Trial and error

The learners could draw different possible models of the vegetable garden with a perimeter of 24 metres. Different possibilities are:



From the drawings it looks as if the $6\ m \times 6\ m$ square yields the maximum area. But can we be sure?

Strategy 2: Using a table

The drawings above illustrate a variety of ways in which the garden can be fenced. We can draw up a table using the above drawings:

Length in m	Width in m	Area in m^2
0	12	0
1	11	11
2	10	20
3	9	27
4	8	32
5	7	35
6	6	36
7	5	35
8	4	32
9	3	27
10	2	20
11	1	11
12	0	0

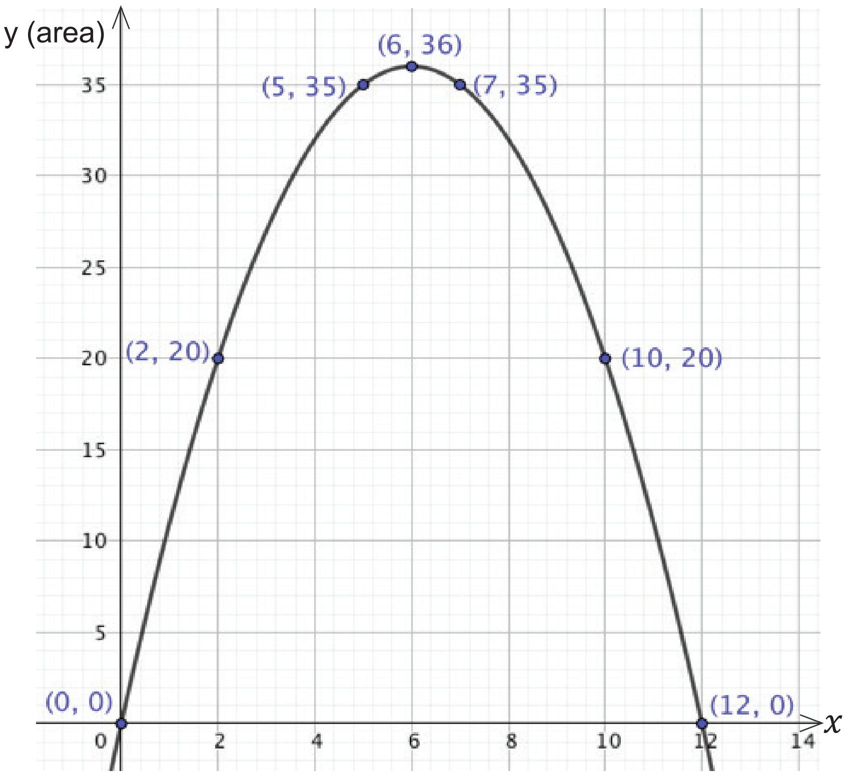
From the table it also looks as if the $6\ m \times 6\ m$ square yields the maximum area. But can we be sure? We used only whole numbers up to now but it is also possible to use real numbers with fraction parts. It may be possible to have a maximum area when x is a real number which is not a whole number.

Strategy 3: Drawing a graph and identify the maximum value

The graph plots x against area using the table. From the table the learners should be able to plot the length on the -axis and the area on the -axis.

x =Length in m	Width	y = Area in m^2
0	12	0
1	11	11
2	10	20
3	9	27
4	8	32
5	7	35
6	6	36
7	5	35
8	4	32
9	3	27
10	2	20
11	1	11
12	0	0

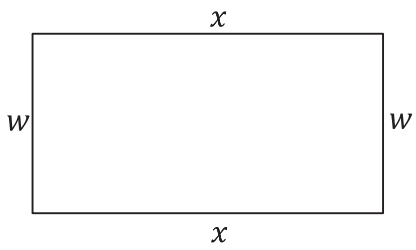
From the table we can conjecture that the maximum area is $36\ m^2$, but to justify the conjecture, we have to look at ALL the values of (not only the integers). The graphical presentation below, which is a parabola, caters for all the real values.



From the table and the graph it looks as if the $6\ m \times 6\ m$ square yields the maximum area. We can now with more confidence conclude that it seems, at least looking at a visual image, that the dimensions yields the maximum area of $36\ m^2$.

Strategy 4: Algebraic methods

Let x = length (horizontal) and w = width (vertical) of the garden (in metres):



We know that the perimeter is 24 metres: $2x + 2w = 24$, then $x + w = 12$. Therefore the width is $-x + 12$.

The area of a rectangle is: Area = length x width = $x(12 - x)$:

Once we have an expression for the area in terms of the side, we can formulate the area as
$y = A(x) = x(12 - x) = 12x - x^2 = -x^2 + 12x.$.

Algebraic method 1: Completing the square using the formula $y = a(x-p)^2 + q$, where q is the minimum or maximum value of the expression.

The area of the vegetable garden is: $A(x) = -x^2 + 12x$

$A(x) = -x^2 + 12x$

$= -(x^2 - 12x)$

$= -\left(x^2 - 12x + \left(-\frac{12}{2}\right)^2 - \left(-\frac{12}{2}\right)^2\right)$

$= -(x^2 - 12x + 36 - 36)$

$= -((x-6)^2 - 36)$

$= -(x-6)^2 + 36$

Therefore, the maximum area is 36 m^2 when $x = 6$. There the length and the width of the garden are 6 metres. We can now conclude with certainty that the maximum area of the garden is 36 m^2 .

Algebraic method 2: Using the formula $x = -\frac{b}{2a}$ to find the value $y = f\left(-\frac{b}{2a}\right)$

The maximum value of the function is reached at the turning point, which lies on the axis of symmetry. So we can use the formula to calculate the maximum area. $f\left(-\frac{b}{2a}\right)$ is the value of f at the turning point.

The area of the vegetable garden is: $A(x) = -x^2 + 12x$. For a maximum area the value $x = -\frac{b}{2a} = -\frac{12}{2(-1)} = 6$, therefore the maximum area is: $y = A\left(-\frac{b}{2a}\right) = A(6) = -36 + 12(6) = 36$

Therefore, the maximum area is 36 m^2 when $x = 6$. There the length and the width of the garden are 6 metres.

***Algebraic method 3:* Using calculus $A'(x) = 0$**

We know that the gradient at the turning point of a function is zero. So we can use the gradient function $A'(x)$ to calculate the -value of the turning point. The area of the vegetable garden is: $A(x) = -x^2 + 12x$

For a maximum area the value $A'(x) = 12 - 2x = 0$

Therefore $x = 6$ and the maximum area is $A(6) = -36 + 12(6) = 36$:

Therefore, the maximum area is 36 m^2 when $x = 6$. There the length and the width of the garden are 6 metres. The maximum area obtained when both length and width of the garden fence is $6\,m$. That is if the garden is in the form of a square.

Considerations for teaching

The teacher should allow learners to grapple with information to find a possible solution to a problem which has been given. The problem posed in this example presents an ideal situation where learners can share and compare their solutions in class. When learners are allowed to express their own ideas, they develop mathematical language and conceptual understanding.

The dimensions of the shape are not necessarily whole numbers and therefore the graph of the parabola can be continuous. This provides a context to discuss number systems.

Learners often think that a square is not a rectangle. They should understand that a square has all the properties of a rectangle (and more!) making it a special rectangle. This example provides an opportunity to show learners that for a fixed perimeter, a square is a rectangle with maximum area.

After the learners' generation of specific examples (drawings) they could create a table with only a limited number of possible values. The learners can compare their findings to gather more values. e.g.

Length in m	**Width in m**	**Area in m^2**
2	10	20
5	7	35
6	6	36
7	5	35
10	2	20

A table will help the learners to organise the data and to find possible patterns as well as a possible maximum value. It will be useful to predict/calculate all the missing whole number values. From the table the learners should be able to plot the length on the horizontal axis and the area on the vertical axis.

x =Length in	**Width**	**y = Area in m^2**
0	12	0
1	11	11
2	10	20
3	9	27
4	8	32
5	7	35
6	6	36
7	5	35
8	4	32
9	3	27
10	2	20
11	1	11
12	0	0

Another more open-ended way of asking this question is to remove the restriction to a rectangular shape. Teachers could ask learners to experiment using different shapes, including circles, to extend the investigation of maximum area given a fixed perimeter.

Learning benefits associated with strategic competence

The learners will be able to make connections between real life contexts, graphical representations, algebraic representations and calculus. The table should enable learners to organise information and detect a pattern (and see the symmetry).

The learners will realise the dimensions of the side-lengths converge to the maximum value for the area when both the length and the width of the garden fence are . That is if the garden is in the form of a square. Using different strategies will give learners the opportunity to think and experience different levels of justification using visual representations, tabular representation, graphical representation and algebraic methods.

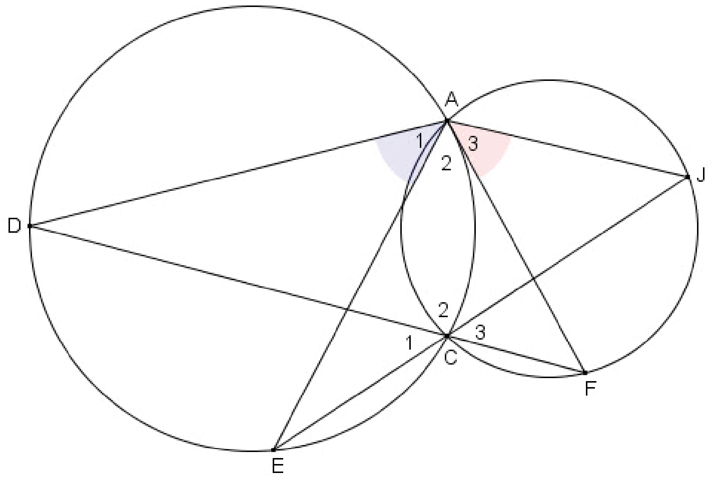
(d) Reasoning

Logical reasoning plays an important role in mathematics. Almost no mathematical problems involve no reasoning at all. Most common is deductive reasoning where one draws conclusions that follow from what is given. This happens often in geometry but also in all mathematics topics. Inductive reasoning happens when the learner has to identify patterns from given data.

Example:

In the diagram point A, C, E, and D, are on one circle and A, C, F and J on another circle. The circles intersect at A and C. The lines EJ and DF intersect in point C.

Prove that $\hat{A}_1 = \hat{A}_3$.



Allow learners time to interpret the information in the diagram. Give learners the opportunity to discuss their interpretations. Read the given information and discuss the possible implications. The accompanying picture only shows one example that satisfies the description.

Proof:

$\hat{A}_1 = \hat{C}_1$ (Angles subtended by a chord at the circle are equal)

but $\hat{C}_1 = \hat{C}_3$ (Vertically opposite angles are equal)

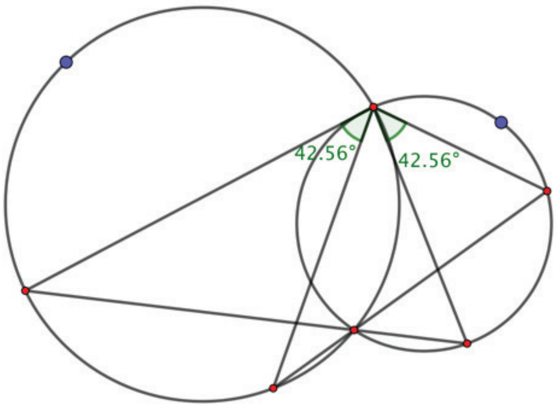
and $\hat{C}_3 = \hat{A}_3$ (Angles subtended by a chord at the circle are equal)

In summary, the chain of reasoning in this case is: $\hat{A}_1 = \hat{C}_1 = \hat{C}_3 = \hat{A}_3$.

Hence $\hat{A}_1 = \hat{A}_3$

Considerations for teaching

Using dynamic geometry software, it is possible to illustrate by dragging different points on the diagram that and are always equal. Although this is not a formal proof it will lay the foundation for justification as well as explanation by deductive proof.



If you accept 'if P then Q' or P $\Rightarrow$ Q as true and you accept P as true, then you must logically accept Q as true. Virtually all mathematical theorems are composed of implications of the type 'if A $\Rightarrow$ B and B $\Rightarrow$ C then A $\Rightarrow$ C'. This valid way of reasoning, which is used to combine two conditionals, is called a 'reasoning chain'. We use reasoning chains like this every day.

In school mathematics (especially in geometry), learners often find it difficult to understand reasoning chains. A real-life example could help them in this regard:

A $\Rightarrow$ B:	If Peter leaves home late, he will miss his train
B $\Rightarrow$ C:	If Peter misses his train, he will be late for work
Then A $\Rightarrow$ C:	If Peter leaves home late, he will be late for work.

All of the links in the chain need to be there, and they need to be correct, for the final statement to be true. Teachers need to give learner ample practice in building up valid reasoning chains.

Achieving the right level of learner competence in deductive reasoning requires great care on the part of teachers. The level of precision necessary to communicate mathematics is high and teachers need to equip their learners to achieve this level of precision.

Learning benefits associated with the ability to reason

Logical reasoning is an important foundational skill in mathematics. Learning mathematics is a sequential process - if you do not fully grasp a certain concept or procedure, you can never hope to understand fully other concepts or procedures that come later and depend upon it.

In the circle geometry example shown above learners are given an opportunity to develop a sense of the deductive nature of mathematics. In order to prove that $\hat{A}_1 = \hat{A}_3$ learners have to identify the previously learnt theorems that they can use to set up a reasoning chain that proves the hypothesis. To do this they need to be able to recall the necessary theorems and state them explicitly.

Disciplined deductive reasoning is crucial for understanding and mathematics.

4. IMPLICATIONS OF TEACHING FOR UNDERSTANDING

The effective implementation of the framework depends on strengthening certain key areas which, if available or properly conceptualised, are collectively the determinants of quality teaching and learning of mathematics. Implications for five key areas are expanded on in this section.

4.1 Curriculum

The Curriculum and Assessment Policy Statement (CAPS) for Mathematics is recognised as the policy that prescribes, *inter alia*, the aims and content specification of mathematics teaching in South Africa.

One of the aims of the curriculum advocates *for deep conceptual understanding in order to make sense of mathematics*. Against this background, one of the key curriculum considerations to enhance conceptual understanding in mathematics is through appropriate sequencing of the topics and concepts that naturally belong together. Fragmentation of topics and concepts creates an impression that they can be taught and learnt in isolation which denies teachers and learners opportunities to build (or recognise) relationships and connections between concepts. The impact of the transition from mother-tongue instruction in the Foundation Phase to English as the LoLT in the Intermediate Phase (and there-after) on curriculum specifications needs to be considered. Furthermore, this framework suggests more time to be spent on conceptual understanding, developing strategic competence and reasoning in mathematics, in contrast with strong focus on procedural fluency that has been the status quo.

For this reason, it is inevitable that curriculum adjustments should be undertaken, where necessary, to reorganise topics and concepts for proper sequencing and to allow time for teaching to all the dimensions of this framework. Since building connections between mathematics concepts and topics should be planned another consideration is to develop curriculum maps which demonstrate the connections between the key topics and concepts in the curriculum. This will not only benefit teachers and learners, but textbooks writers too.

In the FET Phase, a possible solution would be a structural revision of the curriculum to allow learners who want to do university courses that involve mathematics to spend more time on mathematics at school than currently, in order to be better prepared for these numerical courses at university. Not all learners will do mathematics at university level but those who do need to be properly prepared. The framework for teaching for understanding suggests that such learners should be allowed to take an extra mathematics course while at school which needs to be planned as part of the school curriculum. This will have the further advantage that the current mathematics curriculum can be reduced (some content could be moved to the additional subject) to make it more accessible to many learners who are not coping with the large amount of content.

Curriculum review as an implication of the framework also needs to include a review of assessment.

4.2 Assessment

According to the CAPS, assessment is a continuous planned process of identifying, gathering and interpreting information about the performance of learners, using various forms of assessment. It involves four steps:

- generating and collecting evidence of achievement;
- evaluating this evidence;
- recording the findings and
- using this information to understand and thereby assist the learner's development in order to improve the process of learning and teaching.

Traditionally, assessment meant that an assessor should sit with a learner to provide guidance and feedback to the learner however, in 21st century mathematics education; the emphasis on assessment has shifted towards the assessment of the product of learning, also known as summative assessment. In line with the framework, assessment should be more than just summative – it should be undertaken for diagnostic, formative or summative purposes and it should be both informal and formal. Whatever the nature of an assessment, regular feedback should be provided to learners to enhance the learning experience.

Tests and exams should be central experiences in learning, not just something to be done as quickly as possible after teaching has ended in order to produce a final grade. To let learners show what they know and are able to do is a different business from the all too conventional practice of counting learners' errors on questions.

Formative assessment serves as an integral component in the learning process. It can be used as a means for tracking learner progress and ensuring high-stakes test preparedness, but it can also play a significant role in the process of changing instructional practices. The teacher's ability to know what to teach next and how to adapt instruction in the light of evidence is critical to effective formative assessment.

Formative assessment instruments (or tests) should be carefully designed to provide intermittent markers at strategic points in the curriculum implementation. A formative assessment test should be a set of carefully designed questions to address learner misconceptions and a tool to address learning targets. These tests should be aligned with the mathematics curriculum but should also test the critical aspects of a topic, drawing on the basic dimensions of the framework discussed, i.e. conceptual understanding, procedural fluency, own strategies and reasoning.

Critical and extensive engagement with the assessment instruments on the part of the teachers, designed to highlight core mathematics concepts, together with reflective implementation of such tasks, will improve the teaching and learning of mathematics.

All teachers must bear in mind that assessment should be unbiased, fair, transparent, valid and reliable.

4.3 Learning and teaching support material

LTSMs cover a broad range of materials, however, textbooks and manipulatives are perhaps the most valuable for the teaching and learning of mathematics. LTSM supports teachers in the interpretation and application of the curriculum. Any curriculum refinements that arise out of implementation of the framework will necessitate the redevelopment of the national workbooks and all other curriculum support material. The development of detailed teacher guides for use alongside the learner materials should be a top priority. This should be accompanied by a well-led national project that inclusively seeks to develop the educational resource fabric on the ground.

These resources are essential to the implementation of the framework since the teacher does more than intervene, facilitate and create a learning environment – they also need to identify and select good mathematical tasks for learners to do. Selection of meaningful and appropriate tasks requires deep conceptual understanding on the part of the teacher but this selection can be supported by high quality LTSM.

Materials are needed that make such tasks available to teachers.

4.3.1 Textbooks

Textbooks have always been the most commonly used resource in mathematics classes. In the majority of mathematics textbooks however there is little emphasis on concept development and more emphasis on exercises that focus on routine procedures.

The *Mathematics Teaching & Learning Framework for South Africa* has four interdependent dimensions: conceptual development, procedural fluency, reasoning and strategic competence which should all feature in textbooks developed for learners. Teaching mathematics for understanding requires reconceptualization of the textbooks to ensure that the following are taken into account:

Purposeful activities / exercises: the activities/exercises used should be carefully chosen to ensure that they are fit for purpose and contribute meaningfully to spiral teaching. It must be stressed that the four dimensions of the *Mathematics Teaching & Learning Framework* are interdependent and that this interdependence should be reflected in textbooks. Activities should be structured to reflect the four dimensions in an integrated rather than a distinct way.

Unit teaching: to ensure that concepts that naturally belong together are taught together or sequentially. This will assist learners to make connections between mathematical concepts and further strengthen conceptual understanding.

Interpretation the curriculum: one of the key roles of textbooks is to interpret the curriculum correctly and to represent it comprehensively so that reliance on the textbook does not compromise curriculum coverage.

Role of the Teacher's Guide: The Teacher's Guide should not only provide answers of the activities/exercises that are in the learner book (textbook), but should be reconceptualised to emphasise the pedagogical skills necessary to develop all four interdependent dimensions of the *Mathematics Teaching & Learning Framework for South Africa*.

4.3.2 Manipulatives

In primary schools conceptual understanding is enhanced through manipulation of concrete objects. Although teachers are encouraged to be innovative and minimise over reliance on commercially available manipulatives, the basic resource kit consisting of the manipulatives for teachers and learners should be provided. These manipulatives should be carefully chosen to embody or properly represent the concepts they are being used to demonstrate.

4.4 Information and communication technologies (ICT)

In South Africa, the need to incorporate ICT in the classroom has been identified. The South African government positioned itself by drafting a White paper on e-Education which presented the implementation strategies of how ICT will be incorporated in the learning, teaching and administration of all schools. The growth rate of cell phone use in Africa is the highest in the world. In fact, the number of online users in South Africa is about 22.5 million. South Africans access the Internet on their mobile phones since the majority do not have access to computers. Mobile technologies therefore increasingly open new opportunities for education.

There are many powerful free mathematics Apps and software available. They can be categorised in terms of static calculation and graphing software (calculators and graphing calculators) and dynamic software. The importance of ICT notwithstanding, it is worth noting that mathematics teachers, and not ICT tools, are the key to quality education.

The effective use of dynamic software and Apps (e.g. GeoGebra and Desmos) has the potential to strengthen learners' conceptual understanding. In the classroom, dynamic geometry can develop and reinforce concepts, enrich visualisation, and rectify misconceptions. By clicking and dragging the cursor, learners can discover relationships, and formulate and test conjectures. This inductive process lays the foundations for justification and deductive proof.

The use of static calculation and graphing tools gives learners the opportunity to perform quick and accurate calculations and constructions. The effective use of calculators can therefore help learners to focus their efforts on the mathematics concepts that they want to develop, rather than struggling to perform tedious calculations and constructions.

As a result, the use of ICT enables us to create opportunities to work on real-world problems which require complex calculations. It is important to notice that calculators must not be used if the purpose of the lesson it to understand calculations or practice calculations (development of procedural skills). The purpose of the lesson should determine the use of ICT in the mathematics classroom.

4.5 Teacher development

The targets of implementation of the framework are in-service and pre-service teachers, subject advisors, teacher educators and all others involved in the mathematics teaching profession. There is a need for a more coherent approach to teacher development that incorporates in-service and pre-service teacher development at the policy level.

Intensive and systematic teacher development programmes need to be conceptualised and implemented to ensure that every mathematics teacher from Grade R to Grade 12, including pre-service teachers, are prepared to adapt to the new trajectory of teaching mathematics for understanding. The development of teachers' deep knowledge of the mathematical content that they teach should be prioritised since teaching according to the framework is not possible without the necessary content knowledge.

Teacher development should include structured programmes for pre-service teachers and systematic activities and workshops for in-service teachers. Consideration should be given to who develops and implements these programmes. Partnerships with appropriate professional mathematical organisations could be considered to support such development and implementation.

Teachers will need the necessary support to implement the framework. Many teachers view mathematics algorithmically and the framework conceptualises teaching in a different way which will place heavy demands on teachers. Teachers who have never been exposed to the balanced and dynamic type of teaching proposed in the framework would benefit from direct classroom support along with LTSM and assessment materials that promote the appropriate classroom culture.

4.6 Language of learning and teaching (LoLT)

Mathematics teachers need to keep the focus on mathematics *but* this cannot be done without language. Research shows that it is beneficial for learners (throughout the school system) to be able to use their home language together with English when discussing mathematical ideas. It has also been shown that learning in the home language in the Foundation Phase has a positive effect on achievement in the Intermediate Phase, both for language and mathematics.

Researchers argue that in multilingual contexts the emphasis needs to be taken off language and the focus should be on the learning of the mathematical concepts. In a mathematics class, language should be a tool used to enable mathematical conceptual learning. Bearing this in mind, the implementation of the framework will rest heavily on language – both on the part of teachers and learners in the system.

Whatever the language of expression might be, spoken language needs to be used in such a way that learners are able to express their thoughts as clearly as possible, while they grapple with the mathematical concepts that they are learning. The use of language should not interfere with the learners' ability to speak about what they are doing and make conceptual generalisations. The practices of code-switching and (more recently) translanguaging speak about flexible language practices. Both practices have been shown to be beneficial to the learning and teaching of mathematics and contribute to learning for understanding as proposed in the framework.

Materials (LTSM and assessment instruments) need to take into consideration the situation on the ground with respect to language diversity in order to promote understanding. One way of addressing this would be the provision bilingual materials – with English text running parallel with the other ten official languages. This addresses the issue of the development of the mathematical register and the diversity of language in school classrooms without holding back the teaching and learning of mathematics.

5. Recommendations: Implementation of the Mathematics Framework

5.1 Advocacy

5.1.1 The Framework should be mediated with all stakeholders within the Department of Basic Education (DBE), across the provinces and in other key institutions before they undertake work that builds on the framework principles.

5.1.2 The Teacher Development Chief Directorates at DBE and in the Provinces should take the responsibility for the dissemination and implementation of the Mathematics Framework.

5.2 Production of the Framework Document

The following documents should be produced to allow for optimal dissemination of the framework. The print rendition should be completed by the launch date. Other renditions should follow as soon as possible thereafter.

- Print ready rendition and DVDs
- Poster per Phase
- Online rendition with hyperlinks
- Cell phone rendition

5.3 Strengthening Caps

5.3.1 One of the implications of the Framework is that the curriculum will need to undergo a thorough review by a team of experts from the different education sectors. This task should begin as soon as possible, undertaken by a representative team of stakeholders including mathematics experts, teachers, education department representatives, unions and other relevant parties. Communication about the process and opportunities for input from the wider mathematics and education community should be provided for.

5.3.2 The following issues should be considered for each grade:

- The number of content areas and topics
- The scope and depth of topics
- Time allocated per topic
- Coherence and sequencing of topics
- Appropriate balance between conceptual understanding, procedural fluency, reasoning and strategic competence

5.3.3 Since curriculum is fluid and the process of review should be continuous, in anticipation of future curriculum change, a panel of experts should be appointed to review the curriculum on an ongoing basis and make appropriate recommendations for refinement and strengthening.

5.3.4 In the FET Band this panel should seriously consider making provision for more Mathematics in the curriculum for students who want to continue with numerical University study programmes. This may lead to allowing two mathematics subjects (in the package of seven matriculation subjects).

5.4 Assessment

- Assessment is an integral part of the curriculum and it must be aligned to the curriculum. Tasks and activities for assessment will also require attention, as indicated in the implications section of the framework.
- **School Based Assessment** should be designed to address the balance of the dimensions in the framework.
- Exemplars of assessment items that deal with all aspects of the framework should be developed and provided to teachers.

5.5 In-Service Teacher Development

The teachers are central to the successful implementation of the framework pedagogy. Teachers need to be appropriately trained and not just given a few days of orientation to the framework. During the training they should be given opportunities to learn about and come to understand the framework dimensions which need to be based in a learning-centred classroom. Training should include application activities where teachers are given time to develop examples on their own and plan at least one lesson to include two or more dimensions of the framework.

5.5.1 Implementation plans for in-service teacher development for all teachers in all schools need to be drawn up.

5.5.2 The SA Mathematics Foundation, consisting of the professional mathematics organisations AMESA (Mathematics Educationists) and SAMS (Mathematicians) should be invited to support teacher development. The appointed individuals should receive high-quality training before they are required to disseminate the framework and train teachers.

5.5.3 Differentiated strategies may be required for disseminating and implementing the framework in different contexts, e.g. multi-grade classes, teachers in rural areas, etc.

5.5.4 Subject advisors and curriculum managers should be capacitated in the introduction, implementation and monitoring of the Framework. They should also receive high-quality training before they are required to disseminate the framework and train teachers.

5.5.5 Relevant stakeholders (Unions, NGOs and other interested parties) should be brought on board.

5.5.6 Online digital content should be produced to support teachers to undertake development in their own time with regard to all dimensions of the Framework.

5.5.7 All training workshops should be SACE accredited so that there is external quality checking of these courses and teachers will be able to earn CPD points when they complete the courses.

5.6 Pre-Service Teacher Development

Pre-service teachers need to be prepared to teach in the classroom in ways that are aligned with the framework. This requires adequate practical opportunities together with assignments that focus specifically on teaching the mathematics curriculum. The curricula planned for their courses should give explicit attention to the framework and should preferably include at least one module that includes the framework as a reading and working document. The framework could be addressed in methodology courses, become part of assessment, used in planning for the practicum, etc.

5.6.1 HEIs should be deliberately informed about the framework and required to include the four dimensions within a learning-centred classroom in their programmes.

5.6.1 Pre-service teacher training must take into account the framework while addressing content knowledge across curriculum.

5.6.1 Training related to assessment aligned with the framework should be included.

5.6.1 Collaboration with the HRDC Standing Committee for mathematics should be established.

5.7 Learning and Teaching Support Material (LTSM)

LTSM supports teachers in the interpretation and application of the curriculum. Any curriculum refinements that arise out of the implementation of the framework will necessitate the revision of all curriculum support material, including the national mathematics workbooks and the Sasol-Inzalo (Siyavula) materials. Provision of all necessary LTSM (print and other) should be planned and included in the appropriate budgets.

5.7.1 Text books, DBE workbooks and other print material should be developed over time so that high quality appropriate materials are made available to schools. We strongly advise AGAINST the tendency to set unrealistic timeframes for the revision and approval of LTSM materials.

5.7.2 A panel of experts should be appointed to review LTSM on an ongoing basis. There should not be a system of cut-off deadlines that sometimes lead to inappropriate choices of books for publication.

5.7.3 Necessary resources should be supplied to learners in all phases to support teaching and learning for understanding. For example, base ten blocks in FP and IP and a mathematical construction set in SP.

5.7.4 An investigation of learner access to calculators in Quintiles 1 and 2 secondary schools should be undertaken as there appears to be evidence that certain learners are writing the matric exam without a calculator.

5.8 Information and Communication Technologies (ICT)

The need to incorporate ICT in the classroom has been identified. However, all framework implementation initiatives should regard as sacred that mathematics teachers, and not ICT tools, are key to quality education.

5.8.1 National portals should be set up with appropriate e-resources to support teachers in implementing all aspects of the framework

5.8.2 These portals should be continuously updated to offer guidance and support to teachers in the use of online material.

5.8.3 ICT is not a prerequisite for teaching for understanding.

5.9 Language of Learning and Teaching (LOLT)

Language plays an important and critical role in the teaching, learning and understanding of mathematics. Mathematics teachers need to keep the focus on mathematics but this cannot be done without appropriate language. Research shows that it is beneficial for learners (throughout the school system) to be able to use their home language together with English when discussing mathematical ideas.

5.9.1 In the FP, teachers and learners should be free to use their home language together with English when discussing mathematical ideas.

5.9.2 The provision of bilingual materials – with English text running parallel to the LoLT of the school – will provide multilingual access to learners. This should be considered at least for the FP and Grade 4.

5.9.3 Assessment needs to align with the recognition of multilingual resources and be presented in bilingual format.

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- **Foundation (FP) Team:** Dr Roy Venketsamy (convener), (University of Pretoria); Prof Mellony Graven, (Rhodes University), Ms Zanele Mofu, (Eastern Cape Education Department);
- **Intermediate Phase (IP) Team:** Mr Manare Setati (convener), (University of Limpopo), Dr Ronel Paulsen, (Mathematics Education Consultant), Ms Heather Collins, (Masicorp);
- **Senior Phase (SP) Team:** Dr Moeketsi Mosia (convener), (Sol Plaatjie University), Dr David Sekao, (University of Pretoria), Ms Asiya Hendriks, (NECT), Dr Craig Pournara, (University of Witwatersrand);
- **Further Education and Training (FET) Team**: Dr Radley Mahlobo (convener), (Vaal University of Technology), Prof Gerrit Stols, (University of Pretoria), Prof Johann Engelbrecht, (South African Mathematics Foundation);

The **Reference Team** of experts in the field assisted in the finalisation of the Mathematics Framework: Prof Hamsa Venkatakrishnan (Witwatersrand University), Prof Mamokgethi Phakeng (University of Cape Town), Ms Junko Funaki and Mr Okira Oshima (Japanese International Cooperation, JICA), Prof Jill Adler (University of Witwatersrand), Prof Rajendran Govender (University of Western Cape), Dr Phethiwe Matutu (National Research Foundation), Dr Paddy Padayachee, and Ms Marlene Sasman (MST Ministerial Task Team, 2013), Dr JJ Dhlamini (University of South Africa), DBE Specialists, Ms Sindi Mcosana (GET Mathematics), Mr Thomas Masango (Mathematical Literacy), Mr Leonard Mudau (FET Mathematics), Dr Mark Chetty (Assessment), Mr Philip Dikgomo (Teacher Development), and Mathematics Curriculum Specialists from Cuba (Rafael Mora Galán, Miguel Angel, Toboso Cruz, Pedro Pérez Campbell, Ariel Pérez Hechavarría, Yoandy de Armas Ojeda, and Heyner Alberto Osorio Lara).

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6. CONCLUSION

This Framework is based on the well-established premise that a learner cannot be a passive recipient of knowledge. It also acknowledges the role of a teacher as being a mediator of learning, and not the sole source of information, drawing on learners' experience. In some cases by learner experience we mean the concrete physical environmental background of the learner that can enhance mathematical conceptual understanding. In other cases we mean the pre-requisite abstract knowledge that a maturing learner brings into classroom. As in all aspects of life, if learning is grounded on realistic learner experience, mastery of concepts will follow. The Framework also incorporates the out-of-school experience of the learner into the classroom dynamics, so as to maximise the conceptual understanding by the learner.

These are the considerations that gave rise to this framework. The role of the teacher is, among others, to select and present to learners meaningful mathematical tasks, to afford learners opportunity to seek solutions to given problems, and give reasons for their answers. The solution process of the learners positions the teacher to know when and how to intervene.

The Framework proposes that teachers must create a learning environment that facilitates the demystification of mathematics as a difficult subject by bringing it closer to the learner experience. This can be done though teaching for understanding – developing learners' conceptual and procedural knowledge and enabling them to reason effectively and draw on a range of strategies to do mathematics.

7. BIBLIOGRAPHY

The following references were used in the compilation of the framework.

Adler, J. (2001). *Teaching mathematics in multilingual classrooms*. Dordrecht, The Netherlands: Kluwer Academic Press.

Andrew, R. (2017). *Procedural vs conceptual approaches to mathematics - a comparison*. Obtained from: http://www.learnimplementshare.com/procedural-vs-conceptual-math-comparison.html

Askew, M., Rhodes, V., Brown, M., Wiliam, D., & Johnson, D. (1997). *Effective Teachers of Numeracy: Report of a study carried out for the Teacher Training Agency*. London.

Barwell, R., Barton, B., & Setati, M. (2009). Educational Studies in Mathematics, *Multilingual Issues in Mathematics Education.* 64(2), 113-119.

Bellanca, J., & Brandt, R. (Eds.). (2010). *21st century skills: Rethinking how Students Learn.* Solution Tree Press: Bloomington, IN.

Black, P., & Wiliam, D. (1998). Assessment and Classroom Learning, Assessment in Education: Principles, *Policy & Practice*, 5(1), 7-73.

Black, P. (1998). *Testing: friend or foe? Theory and practice of assessment and testing.* London: Falmer Press.

Clarke, D., Cheeseman, J.; Gervasoni, A., Gronn, D., Horne, M., McDonough, A., Montgomery, P; Roche, P., Sullivan, P., Clarke, B., & Rowley, G. (2002). *Early Numeracy Research Project Report.* DEET and Catholic Education Office: Melbourne.

Cobb, P., Stephan, M., McClain, K., & Gravemeijer, K. (2001). Rethinking Methodology in the Learning Sciences. *The Journal of the Learning Sciences*, 10(1), 113-163. Stable URL: http://www.jstor.org/stable/1466831, Accessed: 26-02-2018 12:49 UTC

Department of Basic Education. (2011). *Curriculum and Assessment Policy Statement, Final Draft, Foundation Phase Mathematics Grade R-3.* Pretoria: DBE.

Department of Basic Education. (2011). *National Development Plan 2030. Our Future-make it work.* Pretoria: DBE.

Department of Education. (2003). *White Paper on e-Education. Transforming Learning and Teaching through Information and Communication Technologies.* Pretoria: Department of Education.

Engelbrecht, J. (2010). Adding structure to the transition process to advanced mathematical activity. *International Journal of Mathematical Education in Science and Technology*, 41(2), 143-154. http://www.tandfonline.com/doi/abs/10.1080/00207390903391890

Engelbrecht, J., & Mwambakana, J. (2016). Validity and diagnostic attributes of a mathematics Olympiad for junior high school contestants. *African Journal of Research in Mathematics, Science and Technology Education*, *20*(2), 175–188. http://dx.doi.org/10.1080/18117295.2016.1190211

García, O., & Wei, L. (2014). *Translanguaging: language, bilingualism and education.* Basingstoke, Hampshire ; New York: Palgrave Macmillan.

Geyser, H. (2004). Learning from assessment. In S. Gravett & H. Geyser. (Eds.). *Teaching and learning in higher education* (pp. 90-110). Pretoria, South Africa: Van Schaik.

Graven, M., & Stott, D. (2012). Conceptualising procedural fluency as a spectrum of proficiency. In S. Nieuwoudt, D. Laubscher, & H. Dreyer (Eds.), *Proceedings of 18th Annual National Congress of the Association for Mathematical Education of South Africa (AMESA)* (pp. 146–156). Potchefstroom: North-West University.

Graven, M., Venkat, H., Westaway, L., & Tshesane, H. (2013). Place value without number sense: Exploring the need for mental mathematical skills assessment within the Annual National Assessments. *South African Journal of Childhood Education*, 3(2), 131-144.

Harlen, W., & James, M.J. (1977). Assessment and learning: differences and relationships between formative and summative assessment. *Assessment in Education*, 4(3), 365-380.

Hiebert, J., & Carpenter, Th. P. (1992). Learning and teaching with understanding. In: D. W. Grouws (Ed.). *Handbook of Research in Teaching and Learning of Mathematics* (pp. 65-97). New York: Macmillan.

Holmes, E. E. (1995). *New Directions in Elementary School Mathematics: Interactive Teaching and Learning.* New Jersey: Prentice Hall.

Huntley, B., Engelbrecht, J., & Harding, A. (2009). Can multiple choice questions be successfully used as an assessment format in undergraduate mathematics? *Pythagoras*, 69, 3-16.

Kenderov, P. S. (2006). Competitions and Mathematics Education. *Proceedings of the International Congress of Mathematicians*, Madrid, Spain, (pp. 1583-1598).

Kilpatrick, K., Swafford, J., & Findell, B. (Ed.). (2001). *Adding It Up Helping Children Learn*. Mathematics National Academy Press, Washington D.C.

Lewis, G., Jones, B., & Baker, C. (2012). Translanguaging: origins and development from school to street and beyond. *Educational Research and Evaluation*, 18(7), 641–654. http://doi.org/10.1080/13803611.2012.718488

Moavenzadeh, J. (2015). *The 4th Industrial Revolution: Reshaping the Future of Production*. World Economic Forum.

Moschkovich, J. N. (Ed.). (2010). *Language and mathematics education: multiple perspectives and directions for research.* Charlotte, NC: Information Age Pub.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: NCTM. Available online at http://standards.nctm.org

NCTM. The Math Forum. (2001). *Inductive vs. Deductive Reasoning.* http://mathforum.org/library/drmath/view/55695.html

Perry, M. (1991). Learning and Transfer: Instructional conditions and conceptual change. *Cognitive Development* 6, 449-468.

Rittle-Johnson, B., & Alibali, M.W., (1999). Conceptual and Procedural knowledge of Mathematics: does one lead to the other? *Journal of Educational Psychology* 91(1), 175-189.

Samson, D. (2007). Patterns of Visualisation. *Learning and Teaching Mathematics* No. 5.

Schollar, E. (2008). *Final report of the Primary Mathematics Research Project. Presentation to the Conference What's Working in School Development*, JET Education Services, February 2008, available at http://www.jet.org.za/item.php?i_id=263).

Setati, M. (2012). Mathematics in Multilingual Classrooms in South Africa: From Understanding the Problem to Exploring Possible Solutions. In B. Herbel-Eisenmann et al. (Eds.), Equity in Discourse for Mathematics Education:

Theories, Practices, and Policies, Mathematics Education Library 55, 125-145. DOI 10.1007/978-94-007-2813-4_8

Setati, M., & Adler, J. (2000). Between languages and discourses: language practices in primary multilingual mathematics classrooms in South Africa, *Educational Studies in Mathematics* 43, 243–269.

Steen, L. A. (1999). Assessing assessment. In B. Gold (Ed.). *Assessment practices in undergraduate mathematics* (pp. 1-8). Washington, DC: Mathematical Association of America.

Stiggins, R. J., Arter, J. A., Chappuis, J., & Chappuis, S. (2004). *Classroom assessment for student learning: Doing it right—using it well*. Portland, OR: Assessment Training Institute.

Taylor, S., & von Fintel, M. (2016). Estimating the impact of language of instruction in South African primary schools: A fixed effects approach. *Economics of Education Review*, 50, 75–89.

Van der Nest, Long, C., & Engelbrecht, J. (2018). The impact of formative assessment activities on the development of teacher agency in mathematics teachers. To appear in *South African Journal of Education*.

Vincent, J. & Stacey, K. (2008). Do Mathematics Textbooks Cultivate Shallow Teaching? Applying the TIMSS Video Study Criteria to Australian Eighth-grade Mathematics Textbooks. *Mathematics Education Research Journal, 20*(1), 82–107.

Vlassis, J. (2004). Making sense of the minus sign or becoming flexible in ‘negativity’. *Learning and Instruction*, 14, pp. 469–484.

WEF. (2016). *The Future of Jobs Employment, Skills and Workforce Strategy for the Fourth Industrial Revolution.*

Wits Maths Connect Secondary Project, based on research and development work supported by the South African Research Chairs Initiative of the Department of Science and Technology and National Research Foundation, Grant No. 71218

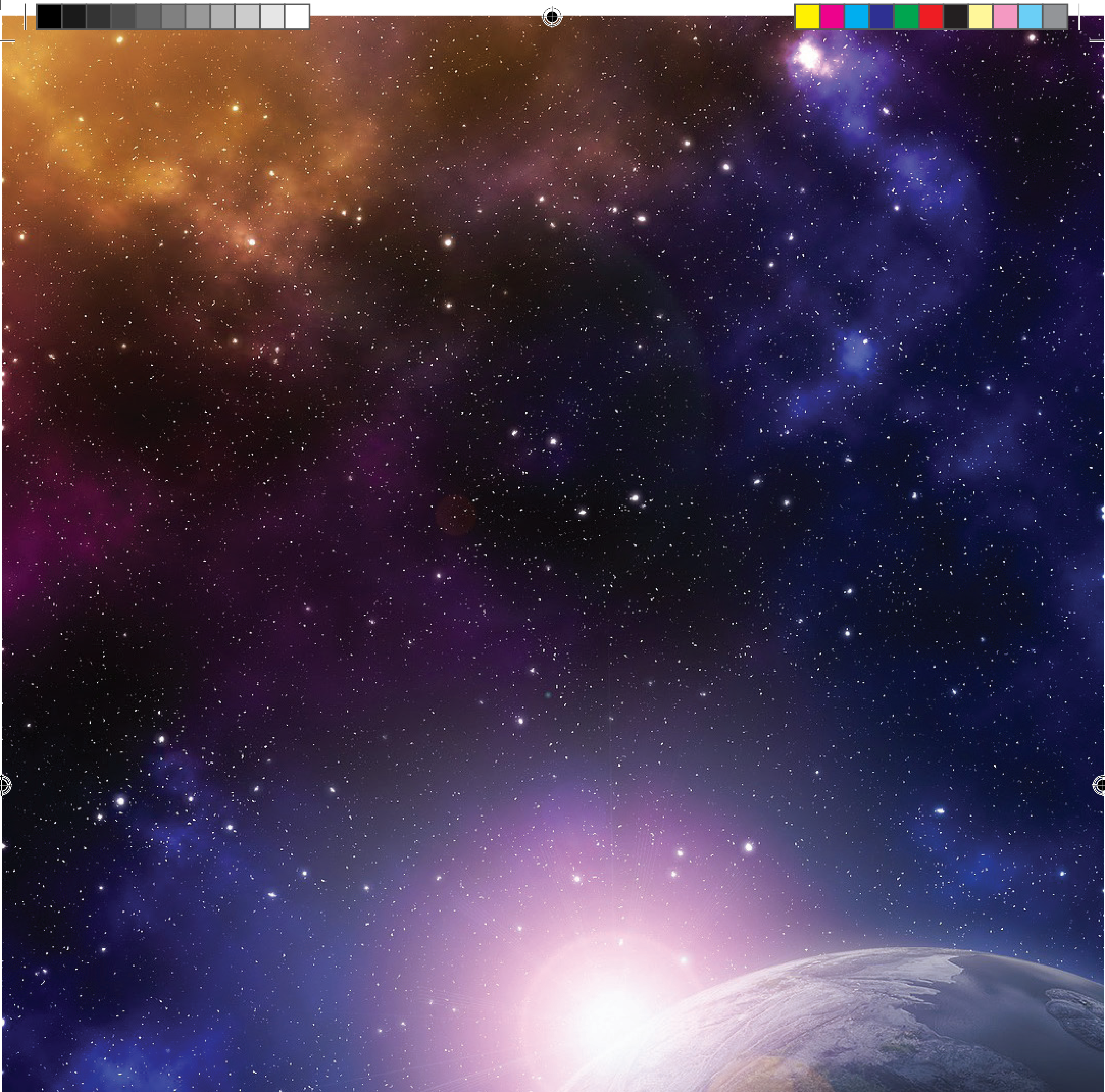
Wright, R. J. (2003). A mathematics recovery: Program of intervention in early number learning. *Australian Journal of Learning Disabilities, 8*(4), 6-11.

Wright, R. J., Martland, J., Stafford, A. K., & Stanger, G. (2006). *Teaching Number: Advancing children’s skills and strategies* (2nd ed., p. 244). London: Paul Chapman Publishing Ltd.

Wright, R.J., Ellemor-Collins, D., & Lewis, G. (2007). Developing pedagogical tools for intervention: Approach, methodology, and an experimental framework. *Mathematics: Essential Research, Essential Practice, 2,* 843-851.

Wright, R.J., Ellemor-Collins, D., & Tabor, P. D. (2012). *Developing Number Knowledge: Assessment, Teaching & Intervention with 7-11-Year-Olds*. London: SAGE.

Zakaria, E., & Syamaun, M. (2017).The Effect of Realistic Mathematics Education Approach on Students’ Achievement And Attitudes Towards Mathematics, *Mathematics Education Trends and Research*, 1, 32-40. doi: 10.5899/2017/metr-00093



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